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Semileptonic B_c-meson decays in sum rules of QCD and NRQCD.

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Abstract

The semileptonic B_c-meson decays into the heavy quarkonia $J/\psi(\eta_c)$ and a pair of leptons are investigated in the framework of three-point sum rules of QCD and NRQCD. Calculations of analytical expressions for the spectral densities of QCD and NRQCD correlators with account for the Coulomb-like α_s/v terms are presented. The generalized relations due to the spin symmetry of NRQCD for the form factors of $B_c \to J/\psi(\eta_c)l\nu_l$ transitions with l denoting one of the leptons e, μ or τ, are derived at the recoil momentum close to zero. This allows one to express all NRQCD form factors through a single universal quantity, an analogue of Isgur-Wise function at the maximal invariant mass of lepton pair. The gluon condensate corrections to three-point functions are calculated both in full QCD in the Borel transform scheme and in NRQCD in the moment scheme. This enlarges the parameteric stability region of sum rule method, that makes the results of the approach to be more reliable. Numerical estimates of widths for the transitions of $B_c \to J/\psi(\eta_c)l\nu_l$ are presented.

1 Introduction

Recently, the CDF collaboration reported on the first experimental observation of the B_c meson, the heavy quarkonium with the mixed heavy flavour [1]. This meson stands among the families of charmonium $\bar{c}c$ and bottominum $\bar{b}b$ in what concerns its spectroscopic properties: two heavy quarks move nonrelativistically, since the confinement scale, determining the presence of light degrees of freedom (sea of gluons and quarks), is suppressed with respect to the heavy quark masses m_Q as well as the Coulomb-like exchanges result in transfers about $\alpha_s m_Q^2$, which is again much less than the heavy quark mass. So, the nonrelativistic picture of binding the quarks leads to the well-known arrangement of system levels, which is very similar for the families mentioned above. The calculations of $\bar{b}c$-mass spectrum were reviewed in [2, 3]. So, we expect

$$M_{B_c} = 6.25 \pm 0.03 \text{ GeV},$$

when the measurement gave

$$M_{B_c}^{exp} = 6.40 \pm 0.39 \pm 0.13 \text{ GeV}.$$

There is an essential difference in the production mechanisms of heavy quarkonia $\bar{c}c$, $\bar{b}b$ and B_c. To bind the $\bar{b}$ and c quarks, one has to produce four heavy quarks in the flavour conserving interactions[1], which allows one to use the perturbative QCD, since the virtualities are determined by the scale of heavy quarks mass. Thus, we see that the production of B_c is relatively suppressed $\sigma(B_c)/\sigma(\bar{b}b) \sim 10^{-3}$ because of the additional heavy quark pair in final states. The basic peculiarities of production mechanisms appear due to the higher orders of QCD even in the leading approximation: the fragmentation regime at high transverse momenta much greater than the quark masses and a strong role of nonfragmentational contributions at $p_T \sim m_Q$, which can be exactly calculated perturbatively; a negligible contribution of octet mechanism [4] because there is no enhancement due to a lower order in α_s. The predictions for the cross sections and distributions of B_c in various interactions are discussed in [5], where we see a good agreement with the meausrements of CDF [1].

In contrast to $\bar{c}c$ and $\bar{b}b$ states decaying due to the annihilation into the light quarks and gluons, the B_c meson is a long lived particle, since it decays due to the weak interaction. The lifetime and various modes of decays were analyzed in the framework of a) potential models [6], b) technique of Operator Product Expansion in the effective theory of NRQCD [7], considering the series in both a small relative velocity v of heavy quarks inside the meson and the inverse heavy quark mass [8], c) QCD sum rules [9, 10] applied to the three-point correlators [11, 12, 13]. The results of potential models and NRQCD are in agreement with each other. So, we expect that the total lifetime is equal to

$$\tau_{B_c} = 0.55 \pm 0.15 \text{ ps},$$

which agrees with the experimental value given by CDF [1]:

$$\tau_{B_c}^{exp} = 0.46^{+0.18}_{-0.16} \pm 0.03 \text{ ps},$$

within the accuracy available.

[1]We do not consider for the production in weak interactions.

Further, the consideration of exclusive B_c decays in the framework of QCD sum rules indicated that the role of Coulomb corrections to the bare quark loop results could be very important to reach the agreement with the other approaches mentioned [13]. This requires working out the α_s/v corrections in NRQCD, which possesses a spin symmetry providing some relations between the exclusive form factors. For the semileptonic decays, such a relation was derived in [14]. Note, that the CDF Collaboration observed 20 events of $B_c^+ \to \psi e^+(\mu^+)\nu$, so that the consistent calculation of semileptonic decay modes is of interest. For theoretical reviews on the B_c meson physics, see [15].

In this paper we perform a detailed analysis of semileptonic B_c decays in the framework of sum rules in QCD and NRQCD. We recalculate the double spectral densities available previously in [11] in full QCD for the massless leptons and add the analytical expressions for the form factors necessary in evaluation of decays to massive leptons and P-wave levels of quarkonium with different quark masses. We analyze the NRQCD sum rules for the three-point correlators for the first time. We derive generalized relations between the NRQCD form factors, which extends the consideration in [14], because we explore a soft limit of recoil momentum close to zero, wherein the velocities of initial and final heavy quarkonia $v_{1,2}$ are not equal to each other, when their product tends to zero, in contrast to the hard limit $v_1 = v_2$. The spin symmetry relations between the form factors are conserved after the taking into account the Coulomb α_s/v corrections, which can be written down in covariant form. We investigate numerical estimates in the sum rules schemes of spectral density moments and Borel transform and show an important role of Coulomb corrections. Next, we perform the calculation of gluon condensate contribution to the three-point sum rules of both full QCD and NRQCD for the case of three massive quarks, for the first time.

The paper is organized as follows. The QCD sum rules of three-point correlators are considered in Section 2, where the spectral densities are calculated in the bare quark-loop approximation and with account of the Coulomb corrections, and the gluon condensate term in the Borel transform scheme is presented. Numerical estimates of semileptonic decay modes are also given here. Section 3 is devoted to the NRQCD sum rules for recoil close to zero. The spin symmetry relations are derived and the gluon condensate is taken into account in the scheme of moments. The results are summarized in Section 4. Appendices A and B contain technical details in evaluation of decay widths for the massive leptons and gluon condensate in full QCD, respectively.

2 Three-point QCD sum rules

In this paper we will use the approach of three-point QCD sum rules [9, 10] in the study of form factors and decay rates for the transitions $B_c^+ \to \psi(\eta_c) l^+ \nu_l$, where l denotes one of the leptons e, μ or τ. This procedure is similar to that of two-point sum rules and the information from the latter on the coupling of mesons to their currents is required in order to extract the values for the form factors. Thus, in our work we will use the meson couplings, defined by the following equations:

$$\langle 0|\bar{q}_1 i\gamma_5 q_2|P(p)\rangle = \frac{f_P M_P^2}{m_1 + m_2}, \tag{1}$$

and

$$\langle 0|\bar{q}_1\gamma_\mu q_2|V(p,\epsilon)\rangle = i\epsilon_\mu M_V f_V, \tag{2}$$

where P and V represent the scalar and vector mesons with desired flavour quantum numbers, respectively, and m_1, m_2 are the quark masses. Now we would like to describe the method used.

2.1 Description of the method

As we have already said, for the calculation of hadronic matrix elements relevant to the semileptonic B_c-decays into the pseudoscalar and vector mesons in the framework of QCD, we explore the QCD sum rule method. The hadronic matrix elements for the transition $B_c^+ \to \psi(\eta_c)l^+\nu_l$ can be written down as follows:

$$\begin{aligned}
\langle\eta_c(p_2)|V_\mu|B_c(p_1)\rangle &= f_+(p_1+p_2)_\mu + f_-q_\mu, & (3)\\
\frac{1}{i}\langle J/\psi(p_2)|V_\mu|B_c(p_1)\rangle &= iF_V\epsilon_{\mu\nu\alpha\beta}\epsilon^{*\nu}(p_1+p_2)^\alpha q^\beta, & (4)\\
\frac{1}{i}\langle J/\psi(p_2)|A_\mu|B_c(p_1)\rangle &= F_0^A\epsilon_\mu^* + F_+^A(\epsilon^*\cdot p_1)(p_1+p_2)_\mu + F_-^A(\epsilon^*\cdot p_1)q_\mu, & (5)
\end{aligned}$$

where $q_\mu = (p_1-p_2)_\mu$ and $\epsilon^\mu = \epsilon^\mu(p_2)$ is the polarization vector of J/ψ-meson. V_μ and A_μ are the flavour changing vector and axial electroweak currents. The form factors $f_\pm, F_V, F_0^A$ and $F_\pm^A$ are functions of q^2 only. It should be noted, that by virtue of transversality of the lepton current $l_\mu = l\gamma_\mu(1+\gamma_5)\nu_l$ in the limit $m_l \to 0$, the probabilities of semileptonic decays into $e^+\nu_e$ and $\mu^+\nu_\mu$ are independent of f_- and F_-^A. Thus, in calculation of these particular decay modes of B_c-meson these form factors can be consistently neglected [11, 6, 13]. However, since the calculation of both the semileptonic decay modes, including e, μ or τ, and some hadronic decays, stands among the goals of this paper, we will present the results for the complete set of form factors given in Eqs. (3)-(5).

Following the standard procedure for the evaluation of form factors in the framework of QCD sum rules, we consider the three-point functions:

$$\begin{aligned}
\Pi_\mu(p_1,p_2,q^2) &= i^2\int dxdy e^{i(p_2\cdot x - p_1\cdot y)}\cdot\\
&\quad \langle 0|T\{\bar{q}_1(x)\gamma_5 q_2(x), V_\mu(0), \bar{b}(y)\gamma_5 c(y)\}|0\rangle, & (6)\\
\Pi_{\mu\nu}^{V,A}(p_1,p_2,q^2) &= i^2\int dxdy e^{i(p_2\cdot x - p_1\cdot y)}\cdot\\
&\quad \langle 0|T\{\bar{q}_1(x)\gamma_\mu q_2(x), J_\mu^{V,A}(0), \bar{b}(y)\gamma_5 c(y)\}|0\rangle, & (7)
\end{aligned}$$

where $\bar{q}_1(x)\gamma_5 q_2(x)$ and $\bar{q}_1(x)\gamma_\nu q_2(x)$ are interpolating currents for states with the quantum numbers of η_c and J/ψ, correspondingly. $J_\mu^{V,A}$ are the currents V_μ and A_μ of relevance to the various cases.

The Lorentz structures in the correlators can be written down as:

$$\begin{aligned}
\Pi_\mu &= \Pi_+(p_1+p_2)_\mu + \Pi_-q_\mu, & (8)\\
\Pi_{\mu\nu}^V &= i\Pi_V\epsilon_{\mu\nu\alpha\beta}p_2^\alpha p_1^\beta, & (9)\\
\Pi_{\mu\nu}^A &= \Pi_0^A g_{\mu\nu} + \Pi_1^A p_{2,\mu}p_{1,\nu} + \Pi_2^A p_{1,\mu}p_{1,\nu} + \Pi_3^A p_{2,\mu}p_{2,\nu} + \Pi_4^A p_{1,\mu}p_{2,\nu}. & (10)
\end{aligned}$$

The form factors $f_\pm$, f_V, F_0^A and $F_\pm^A$ will be determined, respectively, from the amplitudes $\Pi_\pm$, Π_V, Π_0^A and $\Pi_\pm^A = \frac{1}{2}(\Pi_1^A \pm \Pi_2^A)$. In (8)-(10) the scalar amplitudes Π_i are the functions of kinematical invariants, i.e. $\Pi_i = \Pi_i(p_1^2, p_2^2, q^2)$.

To calculate the QCD expression for the three-point correlators we employ the Operator Product Expansion (OPE) for the T-ordered product of currents in (6)-(7). The vacuum correlations of heavy quarks are related to their contribution to the gluon operators. For example, for the $\langle \bar{Q}Q \rangle$ and $\langle \bar{Q}GQ \rangle$ condensates the heavy quark expansion gives

$$\begin{aligned}
\langle \bar{Q}Q \rangle &= -\frac{1}{12 m_Q}\frac{\alpha_s}{\pi}\langle G^2 \rangle - \frac{1}{360 m_Q^3}\frac{\alpha_s}{\pi}\langle G^3 \rangle + ... \\
\langle \bar{Q}GQ \rangle &= \frac{m_Q}{2}\log(m_Q^2)\frac{\alpha_s}{\pi}\langle G^2 \rangle - \frac{1}{12 m_Q}\frac{\alpha_s}{\pi}\langle G^3 \rangle + ...
\end{aligned}$$

Then, in the lowest order for the energy dimension of operators the only nonperturbative correction comes from the gluon condensate:

$$\Pi_i(p_1^2, p_2^2, q^2) = \Pi_i^{pert}(p_1^2, p_2^2, q^2) + \Pi_i^{G^2}(p_1^2, p_2^2, q^2)\langle \frac{\alpha_s}{\pi} G^2 \rangle. \tag{11}$$

The leading QCD term is a triangle quark loop diagram, for which we can write down the double dispersion representation at $q^2 \le 0$:

$$\Pi_i^{pert}(p_1^2, p_2^2, q^2) = -\frac{1}{(2\pi)^2}\int \frac{\rho_i^{pert}(s_1, s_2, Q^2)}{(s_1 - p_1^2)(s_2 - p_2^2)} ds_1 ds_2 + \text{subtractions}, \tag{12}$$

where $Q^2 = -q^2 \ge 0$. The integration region in (12) is determined by the condition

$$-1 < \frac{2 s_1 s_2 + (s_1 + s_2 - q^2)(m_b^2 - m_c^2 - s_1)}{\lambda^{1/2}(s_1, s_2, q^2)\lambda^{1/2}(m_c^2, s_1, m_b^2)} < 1, \tag{13}$$

and

$$\lambda(x_1, x_2, x_3) = (x_1 + x_2 - x_3)^2 - 4 x_1 x_2.$$

The calculation of spectral densities $\rho_i^{pert}(s_1, s_2, Q^2)$ and gluon condensate contribution to (11) will be considered in underlying sections. Now let us proceed with the physical part of three-point sum rules. The connection to hadrons in the framework of QCD sum rules is obtained by matching the resulting QCD expressions of current correlators with the spectral representation, derived from a double dispersion relation at $q^2 \le 0$.

$$\Pi_i(p_1^2, p_2^2, q^2) = -\frac{1}{(2\pi)^2}\int \frac{\rho_i^{phys}(s_1, s_2, Q^2)}{(s_1 - p_1^2)(s_2 - p_2^2)} ds_1 ds_2 + \text{subtractions}. \tag{14}$$

Assuming that the dispersion relation (14) is well convergent, the physical spectral functions are generally saturated by the lowest lying hadronic states plus a continuum starting at some effective thresholds s_1^{th} and s_2^{th}:

$$\begin{aligned}
\rho_i^{phys}(s_1, s_2, Q^2) &= \rho_i^{res}(s_1, s_2, Q^2) + \\
&\quad \theta(s_1 - s_1^{th}) \cdot \theta(s_2 - s_2^{th}) \cdot \rho_i^{cont}(s_1, s_2, Q^2),
\end{aligned} \tag{15}$$

where

$$\rho_i^{res}(s_1, s_2, Q^2) = \langle 0|\bar{c}\gamma_\mu(\gamma_5)c|J/\psi(\eta_c)\rangle\langle J/\psi(\eta_c)|F_i(Q^2)|B_c\rangle\langle B_c|\bar{b}\gamma_5 c|0)\rangle \cdot (2\pi)^2\delta(s_1 - M_1^2)\delta(s_2 - M_2^2) + \text{higher state contributions}, \quad (16)$$

where $M_{1,2}$ denote the masses of quarkonia in the initial and final states. The continuum of higher states is modelled by the perturbative absorptive part of Π_i, i.e. by ρ_i. Then, the expressions for the form factors F_i can be derived by equating the representations for the three-point functions Π_i in (11) and (14), which means the formulation of sum rules.

2.2 Calculating the spectral densities

In this section we present the analytical expressions to one loop approximation for the perturbative spectral functions. We have recalculated their values, already available in the literature [11]. Among new results there are the expressions for ρ_-, ρ_-^A and $\rho_\pm^{'A}$, where $\rho_\pm^{'A}$ are spectral functions, which come from the double dispersion representation of $\Pi_\pm^{'A} = \frac{1}{2}(\Pi_3^A \pm \Pi_4^A)$. These spectral densities are not required for the purposes of this paper, but they will be useful for calculation of form factors for the transition of B_c-meson into a scalar meson[2]. The procedure of evaluating the spectral functions involves the standard use of Cutkosky rules [16]. There is, however, one subtle point in using these rules. At $Q^2 < 0$ there is no problem in applying the Cutkosky rules in order to determine $\rho_i(s_1, s_2, Q^2)$ and the limits of integration over s_1, s_2. At $Q^2 > 0$, which is the physical region, non-Landau-type singularities appear [17, 18], what makes the determination of spectral functions to be quite complicated. In our case we restrict the region of integration in s_1 and s_2 by s_1^{th} and s_2^{th}, so that at moderate values of Q^2 the non-Landau singularities do not contribute to the values of spectral functions. For spectral densities $\rho_i(s_1, s_2, Q^2)$ we have the following expressions:

$$\begin{aligned}
\rho_+(s_1, s_2, Q^2) &= \frac{3}{2k^{3/2}}\{\frac{k}{2}(\Delta_1 + \Delta_2) - k[m_3(m_3 - m_1) + m_3(m_3 - m_2)] - \\
&\quad [2(s_2\Delta_1 + s_1\Delta_2) - u(\Delta_1 + \Delta_2)] \\
&\quad \cdot[m_3^2 - \frac{u}{2} + m_1 m_2 - m_2 m_3 - m_1 m_3]\}, && (17)\\
\rho_-(s_1, s_2, Q^2) &= -\frac{3}{2k^{3/2}}\{\frac{k}{2}(\Delta_1 - \Delta_2) - k[m_3(m_3 - m_1) - m_3(m_3 - m_2)] + \\
&\quad [2(s_2\Delta_1 - s_1\Delta_2) + u(\Delta_1 - \Delta_2)] \\
&\quad \cdot[m_3^2 - \frac{u}{2} + m_1 m_2 - m_2 m_3 - m_1 m_3]\}, && (18)\\
\rho_V(s_1, s_2, Q^2) &= \frac{3}{k^{3/2}}\{(2s_1\Delta_2 - u\Delta_1)(m_3 - m_2) \\
&\quad +(2s_2\Delta_1 - u\Delta_2)(m_3 - m_1) + m_3 k\}, && (19)\\
\rho_0^A(s_1, s_2, Q^2) &= \frac{3}{k^{1/2}}\{(m_1 - m_3)[m_3^2 + \frac{1}{k}(s_1\Delta_2^2 + s_2\Delta_1^2 - u\Delta_1\Delta_2)] \\
&\quad -m_2(m_3^2 - \frac{\Delta_1}{2}) - m_1(m_3^2 - \frac{\Delta_2}{2}) && (20)
\end{aligned}$$

[2] The meson at the P-wave level, for which $\langle 0|\bar{q}_1\gamma_\mu q_2|P(p)\rangle = if_P p_\mu$, where $P(p)$ denotes the scalar P-wave meson under consideration, and $m_1 \neq m_2$.

$$
\begin{aligned}
&\quad +m_3[m_3^2-\frac{1}{2}(\Delta_1+\Delta_2-u)+m_1m_2]\},\\
\rho_+^A(s_1,s_2,Q^2) &= \frac{3}{2k^{3/2}}\{m_1[2s_2\Delta_1-u\Delta_1+4\Delta_1\Delta_2+2\Delta_2^2]\\
&\quad m_1m_3^2[4s_2-2u]+m_2[2s_1\Delta_2-u\Delta_1]-m_3[2(3s_2\Delta_1+s_1\Delta_2)\\
&\quad -u(3\Delta_2+\Delta_1)+k+4\Delta_1\Delta_2+2\Delta_2^2+m_3^2(4s_2-2u)] \qquad (21)\\
&\quad +\frac{6}{k}(m_1-m_3)[4s_1s_2\Delta_1\Delta_2-u(2s_2\Delta_1\Delta_2+s_1\Delta_2^2+s_2\Delta_1^2)\\
&\quad +2s_2(s_1\Delta_2^2+s_2\Delta_1^2)]\},\\
\rho_-^A(s_1,s_2,Q^2) &= -\frac{3}{2k^{5/2}}\{kum_3(2m_1m_3-2m_3^2+u)+12(m_1-m_3)s_2^2\Delta_1^2+\\
&\quad k\Delta_2[(m_1+m_3)u-2s_1(m_2-m_3)]+2\Delta_2^2(k+3us_1)(m_1-m_3)\\
&\quad +\Delta_1[ku(m_2-m_3)+2\Delta_2(k-3u^2)(m_1-m_3)]+ \qquad (22)\\
&\quad 2s_2(m_1-m_3)[2km_3^2-k\Delta_1+3u\Delta_1^2-6u\Delta_1\Delta_2]-\\
&\quad 2s_1s_2(km_3-3\Delta_2^2(m_1-m_3))],\\
\rho_+^{\prime A}(s_1,s_2,Q^2) &= -\frac{3}{2k^{5/2}}\{-2(m_1-m_3)[(k-3us_2)\Delta_1^2+6s_1^2\Delta_2^2]+\\
&\quad ku(m_1-m_3)(2m_3^2+\Delta_2)+ku^2m_3+\Delta_1[ku(2m_1-m_2-3m_3)\\
&\quad -2(m_1-m_3)(ks_2-k\Delta_2+3u^2\Delta_2)]- \qquad (23)\\
&\quad 2s_1[(m_1-m_3)(2km_3^2-6u\Delta_1\Delta_2-3u\Delta_2^2)+\\
&\quad 2s_2(km_3+3m_1\Delta_1^2-3m_3\Delta_1^2)+k\Delta_2(2m_1-m_2-3m_3)]\},\\
\rho_-^{\prime A}(s_1,s_2,Q^2) &= \frac{3}{2k^{5/2}}\{2(m_1-m_3)[(k+3us_2)\Delta_1^2+6s_1^2\Delta_2^2]+\\
&\quad ku(m_1-m_3)(2m_3^2+\Delta_2)+ku^2m_3+\Delta_1[ku(-2m_1-m_2+m_3)\\
&\quad -2(m_1-m_3)(ks_2-k\Delta_2+3u^2\Delta_2)]+ \qquad (24)\\
&\quad 2s_1[(m_1-m_3)(2km_3^2-6u\Delta_1\Delta_2+3u\Delta_2^2)-\\
&\quad 2s_2(km_3-3m_1\Delta_1^2+3m_3\Delta_1^2)+k\Delta_2(2m_1+m_2-m_3)]\}.
\end{aligned}
$$

Here $k=(s_1+s_2+Q^2)^2-4s_1s_2$, $u=s_1+s_2+Q^2$, $\Delta_1=s_1-m_1^2+m_3^2$ and $\Delta_2=s_2-m_2^2+m_3^2$. m_1, m_2 and m_3 are the masses of quark flavours relevant to the various decays, see prescriptions in Fig. 1.

We neglect hard $O(\frac{\alpha_s}{\pi})$ corrections to the triangle diagrams, as they are not available yet. Nevertheless, we expect that their contributions are quite small $\sim 10\%$ and so, taking into account the accuracy of QCD sum rules, the correction will not change drastically our results.

In expressions (12) the integration over s_1 and s_2 is performed in the near-threshold region, where instead of α_s, the expansion should be done in the parameters $(\alpha_s/v_{13(23)})$, with $v_{13(23)}$ meaning the relative velocities of quarks in $(b\bar{c})$ and $(c\bar{c})$ systems. For the heavy quarkonia, where the quark velocities are small, these corrections take an essential role (as it is the case for two-point sum rules [19, 20]). The α_s/v corrections, caused by the Coulomb-like interaction of quarks, are related with the ladder diagrams, shown in Fig. 2. It is well known, that an account of these corrections in two-point sum rules numerically leads to a double-triple multiplication of Born value of spectral density [23, 21].

Now, let us comment the effect of these corrections in the case of three-point sum rules

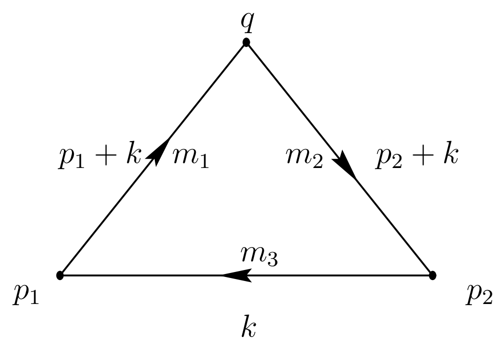

Figure 1: The triangle diagram, giving the leading perturbative term in the OPE expansion of three-point function.

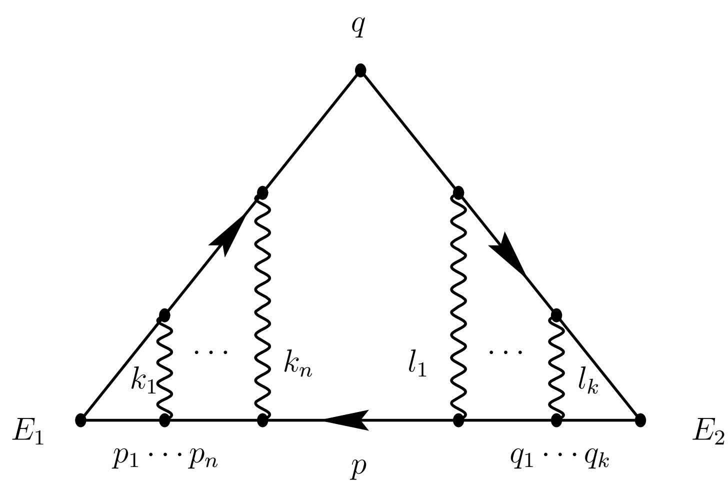

Figure 2: The ladder diagram of the Coulomb-like quark interaction.

[13]. Consider, for example, the three-point function $\Pi_\mu(p_1, p_2, q)$ at $q^2 = q^2_{max}$, where q^2_{max} is the maximum invariant mass of the lepton pair in the decay $B_c \to \eta_c l \nu_l$. Introduce the notations $p_1 \equiv (m_b + m_c + E_1, \vec{0})$ and $p_2 \equiv (2m_c + E_2, \vec{0})$. At $s_1 = M_1^2$ and $s_2 = M_2^2$ we have $E_1 \ll (m_b + m_c)$ and $E_2 \ll 2m_c$. In this kinematics, the quark velocities are small, and, thus, the diagram in Fig. 2 may be considered in the nonrelativistic approximation. We will use the Coulomb gauge, in which the ladder diagrams with the Coulomb-like gluon exchange are dominant. Then the gluon propagator has the form

$$D^{\mu\nu} = i\delta^{\mu 0}\delta^{\nu 0}/\mathbf{k}^2. \tag{25}$$

In this approximation, the nonrelativistic potential of heavy quark interaction in the momentum representation is given by

$$\begin{aligned} \tilde{V}(\mathbf{k}) &= -\frac{4}{3}\alpha_s(\mathbf{k}^2)\frac{4\pi}{\mathbf{k}^2}, \quad \alpha_s(\mathbf{k}^2) = \frac{4\pi}{b_0 ln(\mathbf{k}^2/\Lambda^2)}, \\ b_0 &= 11 - \frac{2}{3}n_f, \quad \Lambda = \Lambda_{\overline{MS}} \exp\left[\frac{1}{b_0}\left(\frac{31}{6} - \frac{5}{9}n_f\right)\right], \end{aligned}$$

with n_f being the number of flavours, while the fermionic propagators, corresponding to either a particle or antiparticle, have the following forms:

$$\begin{aligned} S_F(k+p_i) &= \frac{i(1+\gamma_0)/2}{E_i + k^0 - \frac{|\mathbf{k}|^2}{2m} + i0}, \\ S_F(p) &= \frac{-i(1-\gamma_0)/2}{-k^0 - \frac{|\mathbf{k}|^2}{2m} + i0}. \end{aligned}$$

The notations, concerning Fig. 2, are given by

$$k_i = p_{i+1} - p_i, \quad l_i = q_i - q_{i-1}, \quad p_{n+1} \equiv q_0 \equiv p.$$

Integration over p_i^0, p^0 and q_i^0 by means of residues yields the following expression

$$\begin{aligned} \Pi_\mu(E_1, E_2, q) &= 2N_c g_{\mu 0} \sum_{n=1}^{\infty} \prod_{i=1}^{n} \frac{d\mathbf{p}_i}{(2\pi)^3 (\frac{|\mathbf{p}_i|^2}{2\mu_1} - E_1 - i0)} \tilde{V}((\mathbf{p}_{i+1} - \mathbf{p}_i)^2) \cdot \\ & \frac{1}{\frac{|\mathbf{p}|^2}{2\mu_1} - E_1 - i0} \cdot \frac{1}{\frac{|\mathbf{p}|^2}{2\mu_2} - E_2 - i0} \cdot \\ & \sum_{k=1}^{\infty} \prod_{j=1}^{k} \frac{d\mathbf{q}_j}{(2\pi)^3 (\frac{|\mathbf{q}_j|^2}{2\mu_2} - E_2 - i0)} \tilde{V}((\mathbf{q}_j - \mathbf{q}_{j-1})^2) \frac{d\mathbf{p}}{(2\pi)^3}, \end{aligned} \tag{26}$$

$$\tilde{V}((\mathbf{p}_{n+1} - \mathbf{p}_n)^2) \equiv \tilde{V}((\mathbf{p} - \mathbf{p}_n)^2), \quad \tilde{V}((\mathbf{q}_1 - \mathbf{q}_0)^2) = \tilde{V}((\mathbf{q}_1 - \mathbf{p})^2),$$

where N_c denotes the number of colors, μ_1 and μ_2 are the reduced masses of the $(b\bar{c})$ and $(c\bar{c})$ systems, correspondingly. This three-point function may be expressed in terms of the Green's functions for the relative motion of heavy quarks in the $(b\bar{c})$ and $(c\bar{c})$ systems in the Coulomb field, $G_E^{(i)}(\mathbf{x}, \mathbf{y})$:

$$\begin{aligned} G_E^{(i)}(\mathbf{x}, \mathbf{y}) &= \sum_{n=1}^{\infty} \left(\prod_{k=1}^{n} \int \frac{d\mathbf{p}_k}{(2\pi)^3 (\frac{|\mathbf{p}_k|^2}{2\mu_i} - E_i - i0)} \right) \\ & \times \prod_{k=1}^{n-1} \tilde{V}((\mathbf{p}_k - \mathbf{p}_{k+1})^2) e^{i\mathbf{p}_1 \mathbf{x} - i\mathbf{p}_n \mathbf{y}}. \end{aligned} \tag{27}$$

Comparing the expressions (26) and (27), we find

$$\begin{aligned} \Pi_\mu(E_1, E_2, q^2_{max}) &= 2N_c g_{\mu 0} \int G_{E_1}^{(1)}(\mathbf{x} = 0, \mathbf{p}) G_{E_2}^{(2)}(\mathbf{p}, \mathbf{y} = 0) \frac{d\mathbf{p}}{(2\pi)^3}. \\ &= 2N_c g_{\mu 0} \int G_{E_1}^{(1)}(0, z) G_{E_2}^{(2)}(z, 0) d^3 z. \end{aligned} \tag{28}$$

For the Green's function we use the representation

$$G_E(\mathbf{x}, \mathbf{y}) = \sum_{l,m} \left(\sum_{n=l+1}^{\infty} \frac{\Psi_{nlm}(\mathbf{x}) \Psi^*_{nlm}(\mathbf{y})}{E_{nl} - E - i0} + \int \frac{dk}{(2\pi)} \frac{\Psi_{klm}(\mathbf{x}) \Psi^*_{klm}(\mathbf{y})}{k - E - i0} \right). \tag{29}$$

Provided $\mathbf{x} = 0$, only the terms with $l = 0$ are retained in the sum. Then for the spectral density one has

$$\rho_\mu(E_1, E_2, q^2_{max}) = -2N_c g_{\mu 0} \Psi_1^C(0)\Psi_2^C(0) \int \tilde{\Psi}^C_{1E_1}(\mathbf{p})\tilde{\Psi}^C_{2E_2}(\mathbf{p}) \cdot \frac{d\mathbf{p}}{(2\pi)^3}, \tag{30}$$

where Ψ_i^C are the Coulomb wave functions for the $(b\bar{c})$ or $(c\bar{c})$ systems. An analogous expression can also be derived in the Born approximation:

$$\rho^B_\mu(E_1, E_2, q^2_{max}) = -2N_c g_{\mu 0} \Psi_1^f(0)\Psi_2^f(0) \int \tilde{\Psi}^f_{1E_1}(\mathbf{p})\tilde{\Psi}^f_{2E_2}(\mathbf{p}) \cdot \frac{d\mathbf{p}}{(2\pi)^3}. \tag{31}$$

Here Ψ_i^f stands for the function of free quark motion. Since the continuous spectrum Coulomb functions have the same normalization as the free states, we obtain the approximation

$$\rho_\mu(E_1, E_2, q^2_{max}) \approx \rho^B_\mu(E_1, E_2, q^2_{max}) \frac{\Psi_1^C(0)\Psi_2^C(0)}{\Psi_1^f(0)\Psi_2^f(0)} \equiv \rho^B_\mu(E_1, E_2, q^2_{max})\mathbf{C}, \tag{32}$$

$$\mathbf{C} = \left\{ \frac{4\pi\alpha_s}{3v_{13}} \left[1 - \exp\left(-\frac{4\pi\alpha_s}{3v_{13}}\right)^{-1}\right] \frac{4\pi\alpha_s}{3v_{23}} \left[1 - \exp\left(-\frac{4\pi\alpha_s}{3v_{23}}\right)^{-1}\right]\right\}^{\frac{1}{2}}, \tag{33}$$

where v_{13}, v_{23} are relative velocities in the $(b\bar{c})$ and $(c\bar{c})$ systems, respectively. For them we have the following expressions:

$$v_{13} = \sqrt{1 - \frac{4m_1 m_3}{p_1^2 - (m_1 - m_3)^2}}, \tag{34}$$

$$v_{23} = \sqrt{1 - \frac{4m_2 m_3}{p_2^2 - (m_2 - m_3)^2}}. \tag{35}$$

Eq.(32) is exact for the identical quarkonia in the initial and final states of transition under consideration. However, if the reduced masses are different, then the overlapping of Coulomb functions can deviate from unity, which breaks the exact validity of (32). From a pessimistic viewpoint, this relation can serve as the estimate of upper bound on the form factor at zero recoil. In reality, this boundary is practically saturated, which means that in sum rules at low momenta inside the quarkonia, i.e. in the region of physical resonances, the most essential effect comes from the normalization factor $\mathbf{C}$, determined by the Coulomb function at the origin. The latter renormalizes the coupling constant in the quark-mesonic vertex from the bare value to the "dressed" one. After that, the motion of heavy quarks in the triangle loop is very close to that of free quarks.

In accordance with (8) for the Lorentz decomposition of $\rho_\mu(p_1, p_2, q)$ we have

$$\rho_\mu(p_1, p_2, q^2) = (p_1 + p_2)_\mu \rho_+(q^2) + q_\mu \rho_-(q^2). \tag{36}$$

As we have seen, the nonrelativistic expression of $\rho_\mu(E_1, E_2, q^2_{max})$ is proportional to the vector $(g_{\mu 0})$, which allows us to isolate the evident combination of form factors $f_\pm$. The relations between the form factors appearing in NRQCD at the recoil momentum close to zero will be considered below. Here we stress only that we have

$$\rho_+(q^2_{max}) = \rho^B_+(q^2_{max})\mathbf{C}, \tag{37}$$

where the factor $\mathbf{C}$ has been specified in (33).

In the case of $B_c \to J/\psi l \nu_l$ transition, one can easily obtain an analogous result for $\rho_0^A(q^2)$ (note that the form factor $F_0^A \sim \rho_0^A$ gives the dominant contribution to the width of this decay [14]). In the nonrelativistic approximation we have

$$\rho_0^A(q^2_{max}) = \rho_0^{A,B}(q^2_{max})\mathbf{C}. \tag{38}$$

To conclude this section, note that the derivation of formulas (37) and (38) is purely formal, since the spectral densities are not specified at q^2_{max} (one can easily show $\rho_i^B(q^2)$ to be singular in this point). Therefore, the resultant relations are valid only for q^2 approaching q^2_{max}. Unfortunately, this derivation does not give the q^2 dependence of the factor $\mathbf{C}$. But we suppose that $\mathbf{C}$ does not crucially effect the pole behaviour of the form factors. Therefore, the resultant widths of transitions can be treated as the saturated upper bounds in the QCD sum rules.

2.3 Gluon condensate contribution

In this subsection we will discuss the calculation of Borel transformed Wilson coefficient of the gluon condensate operator for the three-point sum rules with arbitrary masses. The technique used is the same as in [18] with some modifications to simplify the resulting expression. As was noted in [18], this method does not allow for the subtraction of continuum contributions, which, however, only a little change our results as the total contribution of the gluon condensate to the three point sum rule is small by itself ($\leq 10\%$), and, thus, its continuum portion is small, too. The form of the obtained expression does not permit us to use the same argument as in [18] to argue on the absence of those contributions at all. Their argument was based on an expectation, that the typical continuum contribution can show up as incomplete Γ functions in the resulting expression and the absence of them in the final answer leads authors of [18] to the conclusion, that such contributions are actually absent in the processes, they considered.

The gluon condensate contribution to three-point sum rules is given by diagrams, depicted in Fig. 3. For calculations we have used the Fock-Schwinger fixed point gauge [22, 20]:

$$x^\mu A^a_\mu(x) = 0, \tag{39}$$

where A^a_μ, $a = \{1, 2, ..., 8\}$ is the gluon field.

In the evaluation of diagrams in Fig. 3 we encounter integrals of the type[3]

$$I_{\mu_1\mu_2\ldots\mu_n}(a,b,c) = \int \frac{d^4k}{(2\pi)^4} \frac{k_{\mu_1}k_{\mu_2}\ldots k_{\mu_n}}{[k^2 - m_3^2]^a[(p_1+k)^2 - m_1^2]^b[(p_2+k)^2 - m_2^2]^c}. \tag{40}$$

Continuing to Euclidean space-time and employing the Schwinger representation for propagators,

$$\frac{1}{[p^2 + m^2]^a} = \frac{1}{\Gamma(a)} \int_0^\infty d\alpha \alpha^{a-1} e^{-\alpha(p^2+m^2)}, \tag{41}$$

[3]Since the diagrams under consideration do not have UV divergencies, there is no need for a dimensional regularization.

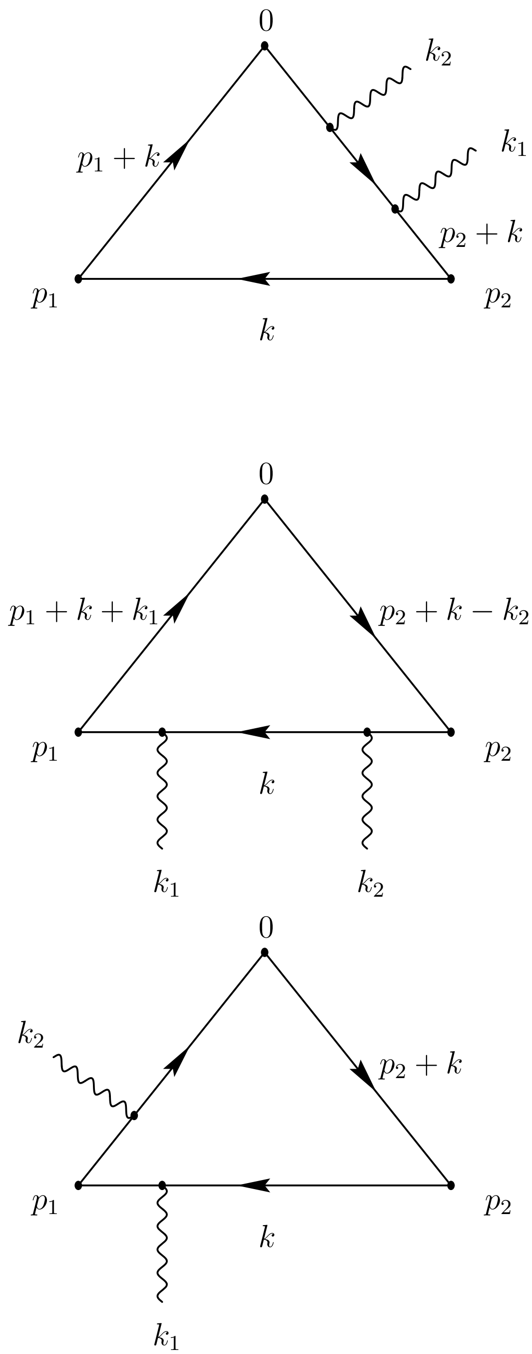

Figure 3: The gluon condensate contribution to three-point QCD sum rules. The directions of p_1, k_1, k_2 momenta are incoming, and that of p_2 is outgoing.

we find the following expression for the scalar integral ($n = 0$):

$$I_0(a,b,c) = \frac{(-1)^{a+b+c}i}{\Gamma(a)\Gamma(b)\Gamma(c)} \int_0^\infty \int_0^\infty \int_0^\infty d\alpha d\beta d\gamma \alpha^{a-1}\beta^{b-1}\gamma^{c-1}$$
$$\int \frac{d^4k}{(2\pi)^4} e^{-\alpha(k^2+m_3^2)-\beta(s_1+k^2+2p_1\cdot k+m_1^2)-\gamma(s_2+k^2+2p_2\cdot k+m_2^2)}. \quad (42)$$

This representation proves to be very convenient for applying the Borel transformation with

$$\hat{B}_{p^2}(M^2)e^{-\alpha p^2} = \delta(1-\alpha M^2). \quad (43)$$

Then, we have

$$\hat{I}_0(a,b,c) = \frac{(-1)^{a+b+c}i}{\Gamma(a)\Gamma(b)\Gamma(c)16\pi^2}(M_{cc}^2)^{2-a-c}(M_{bc}^2)^{2-a-b}\cdot$$
$$U_0(a+b+c-4, 1-c-b), \quad (44)$$

$$\hat{I}_\mu(a,b,c) = \frac{(-1)^{a+b+c+1}i}{\Gamma(a)\Gamma(b)\Gamma(c)16\pi^2}\left(\frac{p_{1\mu}}{M_{bc}^2} + \frac{p_{2\mu}}{M_{cc}^2}\right)(M_{cc}^2)^{3-a-c}(M_{bc}^2)^{3-a-b}\cdot$$
$$U_0(a+b+c-5, 1-c-b), \quad (45)$$

$$\hat{I}_{\mu\nu}(a,b,c) = \frac{(-1)^{a+b+c}i}{\Gamma(a)\Gamma(b)\Gamma(c)16\pi^2}\left(\frac{p_{1\mu}}{M_{bc}^2} + \frac{p_{2\mu}}{M_{cc}^2}\right)\cdot$$
$$\left(\frac{p_{1\nu}}{M_{bc}^2} + \frac{p_{2\nu}}{M_{cc}^2}\right)(M_{cc}^2)^{4-a-c}(M_{bc}^2)^{4-a-b}U_0(a+b+c-6, 1-c-b)$$
$$+\frac{g_{\mu\nu}}{2}\frac{(-1)^{a+b+c+1}i}{\Gamma(a)\Gamma(b)\Gamma(c)16\pi^2}(M_{cc}^2)^{3-a-c}(M_{bc}^2)^{3-a-b}\cdot \quad (46)$$
$$U_0(a+b+c-6, 2-c-b),$$

where M_{bc}^2 and M_{cc}^2 are the Borel parameters in the s_1 and s_2 channels, respectively. Here we have introduced the $U_0(a,b)$ function, which is given by the following expression:

$$U_0(a,b) = \int_0^\infty dy (y + M_{bc}^2 + M_{cc}^2)^a y^b \exp\left[-\frac{B_{-1}}{y} - B_0 - B_1 y\right], \quad (47)$$

where

$$B_{-1} = \frac{1}{M_{cc}^2 M_{bc}^2}(m_2^2 M_{bc}^4 + m_1^2 M_{cc}^4 + M_{cc}^2 M_{bc}^2(m_1^2 + m_2^2 - Q^2)),$$
$$B_0 = \frac{1}{M_{bc}^2 M_{cc}^2}(M_{cc}^2(m_1^2 + m_3^2) + M_{bc}^2(m_2^2 + m_3^2)), \quad (48)$$
$$B_1 = \frac{m_3^2}{M_{bc}^2 M_{cc}^2}.$$

Then, one can express the results of calculation for any diagram in Fig. 3 through $\hat{I}_0(a,b,c)$, $\hat{I}_\mu(a,b,c)$, $\hat{I}_{\mu\nu}(a,b,c)$ and their derivatives over the Borel parameters, using the partial fractioning of the integrand expression together with the following relation:

$$\hat{B}_{p_1^2}(M_{bc}^2)\hat{B}_{p_2^2}(M_{cc}^2)[p_1^2]^m[p_2^2]^n I_{\mu_1\mu_2\ldots\mu_n}(a,b,c) =$$
$$[M_{bc}^2]^m[M_{cc}^2]^n \frac{d^m}{d(M_{bc}^2)^m}\frac{d^n}{d(M_{cc}^2)^n}[M_{bc}^2]^m[M_{cc}^2]^n \hat{I}_{\mu_1\mu_2\ldots\mu_n}(a,b,c). \quad (49)$$

The obtained expression can be further written down in terms of three quantities $\hat{I}_0(1,1,1)$, $\hat{I}_\mu(1,1,1)$, $\hat{I}_{\mu\nu}(1,1,1)$ and their derivatives over the Borel parameters and quark masses by means of

$$\hat{I}_{\mu_1\mu_2\ldots\mu_n}(a,b,c) = \frac{1}{\Gamma(a)\Gamma(b)\Gamma(c)}\frac{d^{a-1}}{d(m_3^2)^{a-1}}\frac{d^{b-1}}{d(m_1^2)^{b-1}}\frac{d^{c-1}}{d(m_2^2)^{c-1}}\hat{I}_{\mu_1\mu_2\ldots\mu_n}(1,1,1). \tag{50}$$

However, contrary to the case, discussed in [18], in such calculations the values of parameters a, b will arise, for which the $U_0(a,b)$-function has no analytical expression (it is connected to nonzero m_3 mass in our case). The analytical approximations for $U_0(a,b)$ at these values of parameters lead to very cumbersome expressions. The search for the most compact form of the final answer leads to the conclusion, that the best decision in this case is to express the result in terms of $U_0(a,b)$-function at different values of its parameters. For this purpose we have used the following transformation properties of $U_0(a,b)$:

$$\begin{aligned}
\frac{dU_0(a,b)}{dM_{bc}^2} &= aU_0(a-1,b) - \left(\frac{m_2^2}{M_{cc}^2} - \frac{m_1^2M_{cc}^2}{M_{bc}^4}\right)U_0(a,b-1) + \\
&\quad \frac{m_1^2+m_3^2}{M_{bc}^4}U_0(a,b) + \frac{m_3^2}{M_{bc}^4M_{cc}^2}U_0(a,b+1), \qquad (51)\\
\frac{dU_0(a,b)}{dM_{cc}^2} &= aU_0(a-1,b) - \left(\frac{m_1^2}{M_{bc}^2} - \frac{m_2^2M_{bc}^2}{M_{cc}^4}\right)U_0(a,b-1) + \\
&\quad \frac{m_2^2+m_3^2}{M_{cc}^4}U_0(a,b) + \frac{m_3^2}{M_{bc}^2M_{cc}^4}U_0(a,b+1). \qquad (52)
\end{aligned}$$

In Appendix B we have presented an analytical expression, obtained in this way, for the Wilson coefficient of gluon condensate operator, contributing to the $\Pi_1 = \Pi_+ + \Pi_-$ amplitude. One can see that, even in this form the obtained results are very cumbersome. So, we have realized the gluon condensate corrections as C++ codes, where the functions $U_0(a,b)$ are evaluated numerically. Analytical approximations, which can be made for the $U_0(a,b)$ functions are discussed in Appendix B.

In Fig. 4 we have shown the effect of gluon condensate on the $f_1(0)$ form factor in the Borel transformed three-point sum rules.

We can draw the conclusion that the calculation of gluon condensate term in full QCD sum rules allows one to enlarge the stability region in the parameter space for the form factors, which indicates the reliability of sum rules technique.

2.4 Numerical results on the form factors

First, we evaluate the form factors in the scheme of spectral density moments. This scheme is not strongly sensitive to the values of threshold energies, determining the region of resonance contribution. In the calculations we put

$$\begin{aligned}
k_{th}(\bar{b}c) &= 1.5 \text{ GeV},\\
k_{th}(\bar{c}c) &= 1.2 \text{ GeV},\\
m_b &= 4.6 \text{ GeV},\\
m_c &= 1.4 \text{ GeV},
\end{aligned} \tag{53}$$

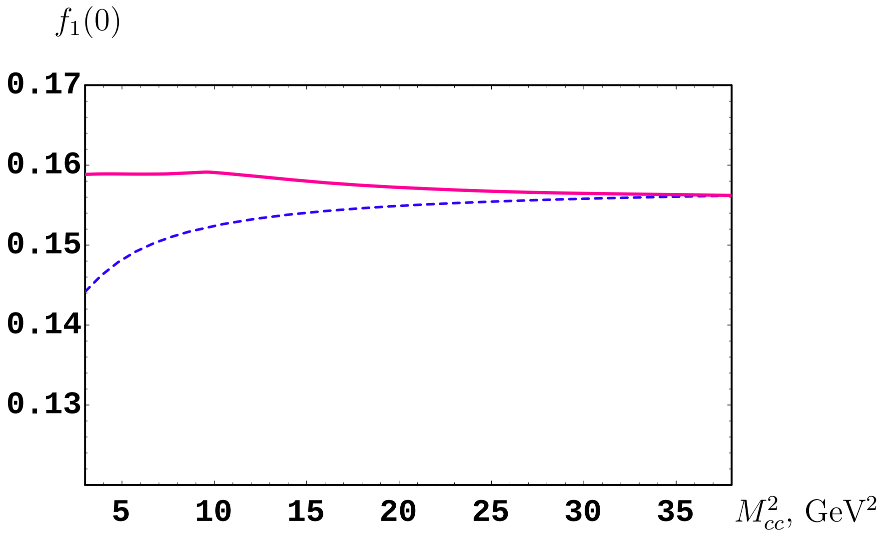

Figure 4: The gluon condensate contribution to the $f_1(0)$ form factor in the Borel-transformed sum rules at fixed $M_{bc}^2 = 70$ GeV^2. The dashed line represent the bare quark-loop results, and the solid curve is the form factor including the gluon condensate term at $\langle \frac{\alpha_s}{\pi} G^2 \rangle = 10^{-2}$ GeV^4.

where k_{th} is the momentum of quark motion in the rest frame of quarkonium. The chosen values of threshold momenta correspond to the minimal energy of heavy meson pairs in specified channels.

The typical behaviour of form factors in the moment scheme of QCD sum rules is presented in Fig. 5.

The evaluation of Coulomb corrections strongly depends on the appropriate set of α_s for the quarkonia under consideration. The corresponding scale of gluon virtuality is determined by quite a low value close to the average momentum transfer in the system. So, the expected α_s is about 0.5. To decrease the uncertainty we consider the contribution of Coulomb rescattering in the two-point sum rules giving the leptonic constants of heavy quarkonia. These sum rules are quite sensitive to the value of strong coupling constant, as the perturbative contribution depends on it linearly. The observed value for the charmonium, $f_\psi \approx 410$ MeV, can be obtained in this technique at $\alpha_s^{coul}(\bar{c}c) = 0.6$. The value $f_{B_c} = 385$ MeV, as it is predicted in QCD sum rules [23], gives $\alpha_s^{coul}(\bar{b}c) = 0.45$. We present the results of the Coulomb enhancement for the form factors in Table 1.
The result after the introduction of the Coulomb correction is shown in Fig. 6. Such large corrections to the form factors should not lead to a confusion, as they are resulted from the fare account of the Coulomb corrections both for bare quark loop diagram and meson coupling constant.

In the scheme of the Borel transformation we find a strong dependence on the thresholds of continuum contribution. We think that this dependence reflects the influence of contributions coming from the excited states. So, the choice of k_{th} values in the same region as in the scheme of spectral density moments results in the form factors, which are approximately 50% greater than the predictions in the moments scheme, where the higher excitations numerically are not

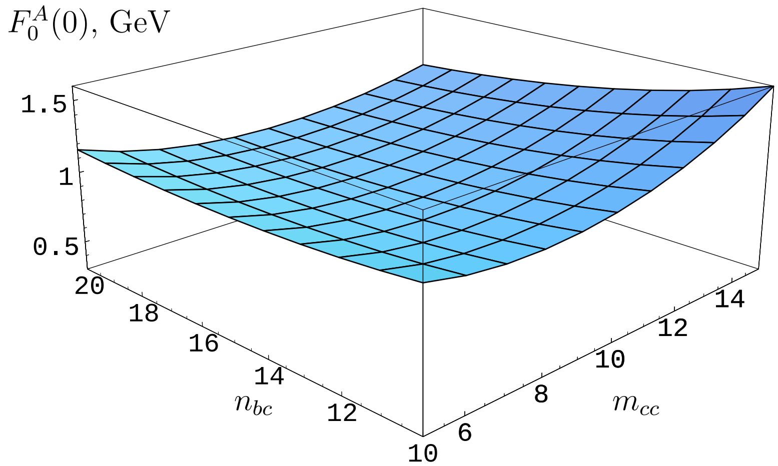

Figure 5: The QCD sum rule results in the moment scheme for the $F_0^A(0)$ form factor in the bare approximation.

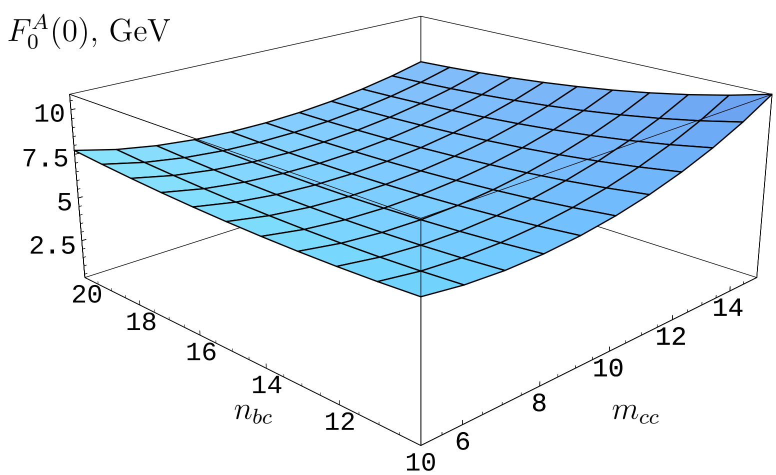

Figure 6: The QCD sum rule results in the moment scheme for the $F_0^A(0)$ form factor with account of Coulomb corrections.

essential. In this case we can explore the ideology of finite energy sum rules [24], wherein the choice of interval for the quark-hadron duality, expressed by means of sum rules, allows one to isolate the contribution of basic states only. So, if we put

$$\begin{aligned} k_{th}(\bar{b}c) &= 1.2 \text{ GeV}, \\ k_{th}(\bar{c}c) &= 0.9 \text{ GeV}, \end{aligned} \tag{54}$$

then the region of the lowest bound states is taken into account in both channels of initial and final states, and the Borel transform scheme leads to the results, which are very close to those of moment scheme. The dependence of calculated values on the Borel parameters is presented in Figs. 7 and 8, in the bare and Coulomb approximations, respectively.

approx.	f_+	f_-	F_V, GeV^{-1}	F_0^A, GeV	F_+^A, GeV^{-1}	F_-^A, GeV^{-1}
bare	0.10	-0.057	0.016	0.90	-0.011	0.018
coul.	0.66	-0.36	0.11	5.9	-0.074	0.12

Table 1: The form factors of B_c^+ decay modes into the heavy quarkonia at $q^2 = 0$ in the bare quark-loop approximation taking into account for the Coulomb correction.

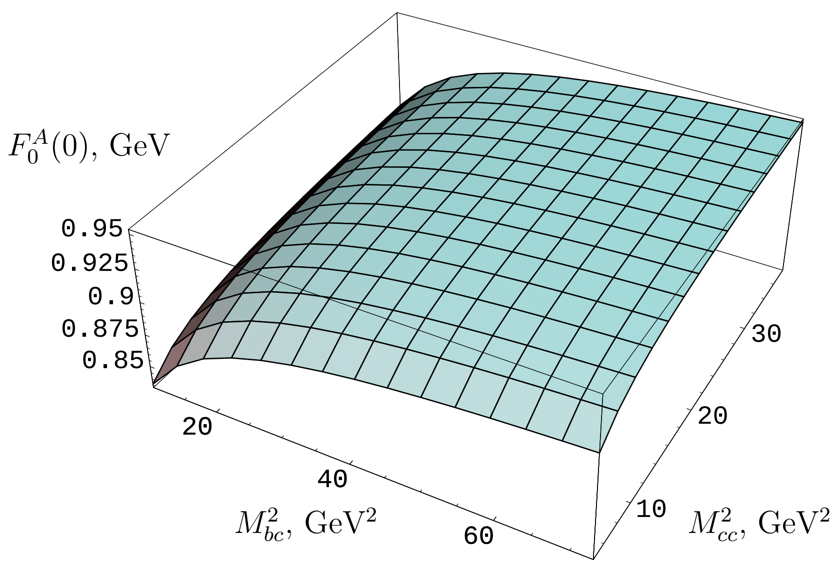

Figure 7: The Borel transformed sum rule results for the $F_0^A(0)$ form factor in the bare approximation of QCD.

As for the dependence of form factors on q^2, the consideration of bare quark loop term

shows that, say, for $F_0^A(q^2)$ it can be approximated by the pole function:

$$F_0^A(q^2) = \frac{F_0^A(0)}{1 - \frac{q^2}{M^2_{pole}}}, \tag{55}$$

with $M_{pole} \approx 4.5$ GeV. The latter is in a good agreement with the value given in [12]. However, we believe that the pole mass can change after the inclusion of α_s corrections[4]. From the naive meson dominance model we expect that $M_{pole} \approx 6.3 - 6.5$ GeV.

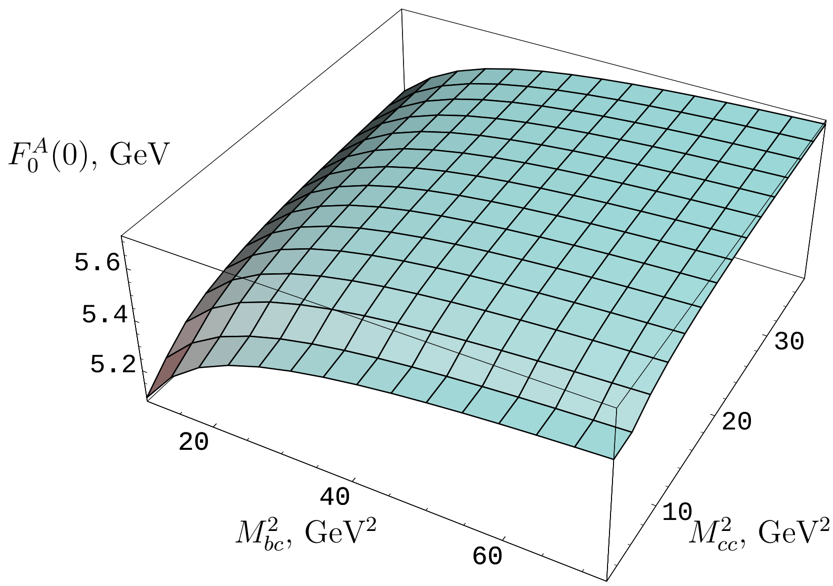

Figure 8: The Borel transformed sum rule results for the $F_0^A(0)$ form factor with account of Coulomb corrections.

We have calculated the total widths of semileptonic decays in the region of $M_{pole} = 4.5-6.5$ GeV, which result in the 30% variation of predictions for the modes with the massless leptons and more sizable dependence for the modes with the τ lepton (see Table 2).

To compare with other estimates we calculate the width of $\bar{b} \to \bar{c}e^+\nu$ transition as the sum of decays into the pseudoscalar and vector states and find[5] $\mathrm{BR}(B_c^+ \to c\bar{c}e^+\nu) \approx 3.4 \pm 0.6$ %, which is in a good agreement with the value obtained in potential models [6] and in OPE calculations [8], where the following estimate was obtained $\mathrm{BR}(B_c^+ \to c\bar{c}e^+\nu) \approx 3.8$ %.

In the presented results we have supposed the quark mixing matrix element $|V_{bc}| = 0.040$.

As for the hadronic decays, in the approach of factorization [25] we assume that the width of transition $B_c^+ \to J/\psi(\eta_c)+$ *light hadrons* can be calculated with the same form factors after the introduction of QCD corrections, which can be easily written down as the factor

[4]In HQET, the slope of Isgur-Wise function acquires a valuable correction due to the α_s-term.

[5]For normalization of the calculated branching ratios we used the total B_c width obtained in the framework of the OPE approach [8].

mode	Γ, 10^{-15} GeV	BR, %
$\eta_c e^+ \nu_e$	11 ± 1	0.9 ± 0.1
$\eta_c \tau^+ \nu_\tau$	3.3 ± 0.9	0.27 ± 0.07
$\psi e^+ \nu_e$	28 ± 5	2.5 ± 0.5
$\psi \tau^+ \nu_\tau$	7 ± 2	0.60 ± 0.15

Table 2: The width of B_c^+ decay modes into the heavy quarkonia and leptonic pair and the branching fractions, calculated at $\tau_{B_c} = 0.55$ ps.

$H = N_c a_1^2$. The factor a_1 represent the hard α_s corrections to the four fermion weak interaction. Numerically we put $a_1 = 1.2$, which yields $H \approx 4.3$. So, we find

$$\begin{aligned} \mathrm{BR}[B_c^+ \to J/\psi + \textit{light hadrons}] &= 11 \pm 2\%, &(56)\\ \mathrm{BR}[B_c^+ \to \eta_c + \textit{light hadrons}] &= 4.0 \pm 0.5\%. &(57) \end{aligned}$$

Neglecting the decays of $\bar{b} \to \bar{c}c\bar{s}$, which are suppressed by both the small phase space and the negative Pauli interference of decay product with the charmed quark in the initial state [8], we evaluate the branching fraction of beauty decays in the total width of B_c as

$$\mathrm{BR}[B_c^+ \to \bar{c}c + X] = 23 \pm 5\%,$$

which is in agreement with the estimates in other approaches [6, 8], where this value is equal to 25 %.

3 Three-point NRQCD sum rules

The formulation of sum rules in NRQCD follows the same lines as in QCD, the only difference is the lagrangian, describing strong interactions of heavy quarks.

3.1 Symmetry of form factors in NRQCD and one-loop approximation

At the recoil momentum close to zero, the heavy quarks in both the initial and final states have small relative velocities, so that the dynamics of heavy quarks is essentially nonrelativistic. This allows us to use the NRQCD approximation in the study of mesonic form factors. As in the case of heavy quark effective theory (HQET), the expansion in the small relative velocities to the leading order leads to various relations between the different form factors. Solving these relations results in the introduction of an universal form factor (an analogue of the Isgur-Wise function) at $q^2 \to q^2_{max}$.

In this subsection we consider the limit

$$\begin{aligned} v_1^\mu &\neq v_2^\mu, &(58)\\ w &= v_1 \cdot v_2 \to 1, \end{aligned}$$

where $v^\mu_{1,2} = p^\mu_{1,2}/\sqrt{p^2_{1,2}}$ are the four-velocities of heavy quarkonia in the initial and final states. The study of region (58) is reasonable enough, because in the rest frame of B_c-meson ($p^\mu_1 = (\sqrt{p^2_1}, \vec{0})$), the four-velocities differ only by a small value $|\vec{p}_2|$ ($p^\mu_2 = (\sqrt{p^2_2}, \vec{p}_2)$, whereas their scalar product w deviates from unity only due to a term, proportional to the square of $|\vec{p}_2|$: $w = \sqrt{1 + \frac{|\vec{p}_2|^2}{p_2^2}} \sim 1 + \frac{1}{2}\frac{|\vec{p}_2|^2}{p_2^2}$. Thus, in the linear approximation at $|\vec{p}_2| \to 0$, relations (58) are valid and take place.

Here we would like to note, that (58) generalizes the investigation of [14], where the case of $v_1 = v_2$ was considered. This condition severely restricts the relations of spin symmetry for the form factors and, as a consequence, it provides a single connection between the form factors.

As can be seen in Fig. 1, since the antiquark line with the mass m_3 is common to the heavy quarkonia, the four-velocity of antiquark can be written down as a linear combination of four-velocities v_1 and v_2:

$$\tilde{v}^\mu_3 = a v^\mu_1 + b v^\mu_2. \tag{59}$$

In the leading order of NRQCD for the kinematical invariants, determining the spin structure of quark propagators in the limit $w \to 1$, we have the following expressions:

$$\begin{aligned} p_1^2 &\to (m_1 + m_3)^2 = \mathcal{M}_1^2, \\ p_2^2 &\to (m_2 + m_3)^2 = \mathcal{M}_2^2, \\ \Delta_1 &\to 2m_3\mathcal{M}_1, \\ \Delta_2 &\to 2m_3\mathcal{M}_2, \\ u &\to 2\mathcal{M}_1\mathcal{M}_2 \cdot w, \\ \lambda(p_1^2, p_2^2, q^2) &\to 4\mathcal{M}_1^2\mathcal{M}_2^2(w^2 - 1). \end{aligned} \tag{60}$$

In this kinematics it is an easy task to show, that in (59) $a = b = -\frac{1}{2}$, i.e.

$$\tilde{v}^\mu_3 = -\frac{1}{2}(v_1 + v_2)^\mu. \tag{61}$$

Applying the momentum conservation in the vertices on Fig. 1 we derive the following formulae for the four-velocities of quarks with the masses m_1 and m_2:

$$\tilde{v}^\mu_1 = v^\mu_1 + \frac{m_3}{2m_1}(v_1 - v_2)^\mu, \tag{62}$$

$$\tilde{v}^\mu_2 = v^\mu_2 + \frac{m_3}{2m_2}(v_2 - v_1)^\mu, \tag{63}$$

and in the limit $w \to 1$, we have $\tilde{v}_1^2 = \tilde{v}_2^2 = 1$, as it should be.

After these definitions have been done, it is straightforward to write down the transition form factor for the current $J_\mu = \bar{Q}_1 \Gamma_\mu Q_2$ with the spin structure $\Gamma_\mu = \{\gamma_\mu, \gamma_5\gamma_\mu\}$

$$\begin{aligned} \langle H_{Q_1\bar{Q}_3}|J_\mu|H_{Q_2\bar{Q}_3}\rangle &= tr[\Gamma_\mu \frac{1}{2}(1 + \tilde{v}^\mu_1\gamma_\mu)\Gamma_1\frac{1}{2}(1 + \tilde{v}^\nu_3\gamma_\nu) \cdot \\ &\quad \Gamma_2\frac{1}{2}(1 + \tilde{v}^\lambda_2\gamma_\lambda)] \cdot h(m_1, m_2, m_3), \end{aligned} \tag{64}$$

where Γ_1 determines the spin state in the heavy quarkonium $Q_1\bar{Q}_3$ (in our case it is pseudoscalar, so that $\Gamma_1 = \gamma_5$), Γ_2 determines the spin wave function of quarkonium in the final state: $\Gamma_2 = \{\gamma_5, \epsilon^\mu\gamma_\mu\}$ for the pseudoscalar and vector states, respectively ($H = P, V$). The quantity h is an universal function at $w \to 1$, independent of the quarkonium spin state. So, for the form factors, discussed in our paper, we have

$$\begin{aligned}
\langle P_{Q_1\bar{Q}_3}|\bar{Q}_1\gamma^\mu Q_3|P_{Q_2\bar{Q}_3}\rangle &= (c_1^P \cdot v_1^\mu + c_2^P \cdot v_2^\mu)\cdot h, &(65)\\
\langle P_{Q_1\bar{Q}_3}|\bar{Q}_1\gamma^\mu Q_3|V_{Q_2\bar{Q}_3}\rangle &= ic_V \cdot \epsilon^{\mu\nu\alpha\beta}\epsilon_\nu v_{1\alpha}v_{2\beta}\cdot h, &(66)\\
\langle P_{Q_1\bar{Q}_3}|\bar{Q}_1\gamma_5\gamma^\mu Q_3|V_{Q_2\bar{Q}_3}\rangle &= (c_\epsilon \cdot \epsilon^\mu + c_1 \cdot v_1^\mu(\epsilon\cdot v_1) + c_2 \cdot v_2^\mu(\epsilon\cdot v_1))\cdot h, &(67)
\end{aligned}$$

where

$$\begin{aligned}
c_\epsilon &= -2,\\
c_1 &= -\frac{m_3(3m_1+m_3)}{4m_1m_2},\\
c_2 &= \frac{1}{4m_1m_2}(4m_1m_2+m_1m_3+2m_2m_3+m_3^2),\\
c_V &= -\frac{1}{2m_1m_2}(2m_1m_2+m_1m_3+m_2m_3), \qquad (68)\\
c_1^P &= 1+\frac{m_3}{2m_1}-\frac{m_3}{2m_2},\\
c_2^P &= 1-\frac{m_3}{2m_1}+\frac{m_3}{2m_2}.
\end{aligned}$$

Then for the form factors in NRQCD we have the following symmetry relations:

$$\begin{aligned}
f_+(c_1^P\cdot\mathcal{M}_2 - c_2^P\mathcal{M}_1) - f_-(c_1^P\cdot\mathcal{M}_2 + c_2^P\cdot\mathcal{M}_1) &= 0,\\
F_0^A\cdot c_V - c_\epsilon\cdot F_V\mathcal{M}_1\mathcal{M}_2 &= 0, \qquad (69)\\
F_0^A(c_1+c_2) - c_\epsilon\mathcal{M}_1(F_+^A(\mathcal{M}_1+\mathcal{M}_2) + F_-^A(\mathcal{M}_1-\mathcal{M}_2)) &= 0,\\
F_0^Ac_1^P + c_\epsilon\cdot\mathcal{M}_1(f_+ + f_-) &= 0.
\end{aligned}$$

Thus, we can claim, that in the approximation of NRQCD, the form factors of weak currents responsible for the transitions between two heavy quarkonium states are given in terms of the single form factor, say, F_0^A. The exception is observed for the form factors F_+^A and F_-^A, since the definite value is only taken by their linear combination $F_+^A(\mathcal{M}_1+\mathcal{M}_2) + F_-^A(\mathcal{M}_1-\mathcal{M}_2)$. This fact has a simple physical explanation. Indeed, the polarization of vector quarkonium ϵ^μ has two components: the longitudinal term ϵ_L^μ and transverse one ϵ_T^μ (i.e. $(\epsilon_T\cdot v_1) = 0$). ϵ_L^μ can be decomposed in terms of v_1^μ and v_2^μ:

$$\epsilon^\mu = \alpha\epsilon_L^\mu + \beta\epsilon_T^\mu, \quad \alpha^2+\beta^2 = 1, \qquad (70)$$

where

$$\begin{aligned}
\epsilon_L^\mu &= \frac{1}{\sqrt{s_2k}}(-2s_2p_1^\mu + up_2^\mu) \to \frac{1}{\sqrt{w^2-1}}(-v_1^\mu + wv_2^\mu),\\
\alpha &= -\frac{2\sqrt{s_2}}{\sqrt{k}}(\epsilon_L\cdot p_1) \to -\frac{1}{\sqrt{w^2-1}}(\epsilon\cdot v_1). \qquad (71)
\end{aligned}$$

From (71) one can see that the decomposition of polarization vector ϵ into the longitudinal and transverse parts in NRQCD is singular in the limit $w \to 1$:

$$\epsilon^\mu = -\frac{1}{w^2-1}(-v_1^\mu + w v_2^\mu)(\epsilon \cdot v_1) + \beta \cdot \epsilon_T^\mu. \tag{72}$$

It means, that the introduction to the form factor F_0^A of an additional term $\Delta F_0^A = (w^2-1)\cdot\delta h$, which vanishes at $w \to 1$ and, thus, is not under control in NRQCD, leads to a finite correction for the form factors F_+^A and F_-^A. This correction is cancelled in the special linear combination of form factors, presented in (69).

In the case of $v_1 = v_2$ we reproduce the single relation between form factors F_0^A and $f_\pm$, as it was obtained early in [14].

Thus, we have obtained the generalized relations due to the spin symmetry of NRQCD lagrangian for the case $v_1 \neq v_2$ in the limit, where the invariant mass of lepton pair takes its maximum value, i.e at the recoil momentum close to zero.

In the one-loop approximation for the three-point NRQCD sum rules, i.e. in the calculation of bare quark loop, the symmetry relations (69) take place already for the double spectral densities ρ_j^{NR} in the limit $|\vec{p}_2| \to 0$. We have checked, that the spectral densities of the full QCD in the NRQCD limit $w \to 1$ satisfy the symmetry relations (69).

It is easily seen that in this approximation,

$$\rho_0^{A,NR} = -\frac{6m_1m_2m_3}{|\vec{p}_2|(m_1+m_3)}. \tag{73}$$

When integrating over the resonance region, we must take into account that

$$\frac{|\omega_2 - \omega_1 \frac{m_{13}}{m_{23}}| m_2}{|\vec{p}_2|\sqrt{2\omega_1 m_{13}}} \leq 1, \tag{74}$$

and so we see, that in the limit $|\vec{p}_2| \to 0$ the integration region tends to a single point. Here $p_1 = (m_1 + m_3 + \omega_1, \vec{0})$, $p_2 = (m_2 + m_3 + \omega_2, \vec{p}_2)$ and $m_{ij} = \frac{m_i m_j}{m_i + m_j}$ is the reduced mass of system $(Q_i\bar{Q}_j)$.

After the substitutions of variables $\omega_1 = \frac{k^2}{2m_{13}}$ and $x = (\omega_2 - \frac{k^2}{2m_{23}}) \cdot \frac{m_2}{|\vec{p}_2|k}$, in the limit $|\vec{p}_2| \to 0$ for the correlator $\Pi_0^{A,NR}$ we have the following expression:

$$\begin{aligned} \Pi_0^{A,NR} &= -\frac{1}{(2\pi)^2}\int \frac{d\tilde{\omega}_1 d\tilde{\omega}_2}{(\tilde{\omega}_1 - \omega_1)(\tilde{\omega}_2 - \omega_2)}\rho_0^{A,NR} = \\ &= \frac{3}{\pi^2}\int_0^{k_{th}} \frac{k^2 dk}{(\omega_1 - \frac{k^2}{2m_{13}})(\omega_2 - \frac{k^2}{2m_{23}})}, \end{aligned} \tag{75}$$

where k_{th} denotes the resonance region boundary. In the method of moments in NRQCD sum rules we put $\omega_1 = -(m_1 + m_3) + q_1$ and $\omega_2 = -(m_2 + m_3) + q_2$, so that in the limit $q_{1,2} \to 0$ we have

$$\begin{aligned} \frac{1}{n!}\frac{1}{m!}\frac{d^{n+m}}{dq_1^n dq_2^m}\Pi_0^{A,NR} &= \Pi_0^{A,NR}[n,m] \\ &= \frac{3}{\pi^2}\int_0^{k_{th}} \frac{k^2 dk}{(\mathcal{M}_1 + \frac{k^2}{2m_{13}})^{n+1}(\mathcal{M}_2 + \frac{k^2}{2m_{23}})^{m+1}}. \end{aligned} \tag{76}$$

In the hadronic part of NRQCD sum rules in the limit $|\vec{p}_2| \to 0$ we model the resonance contribution by the following presentation:

$$\Pi_0^{A,res} = \sum_{i,j} \frac{f_i^{Q_1\bar{Q}_3} M_{1,i}^2}{(m_1+m_3)M_{1,i}^2} \frac{f_j^{Q_2\bar{Q}_3} M_{2,j}}{M_{2,j}^2} F_{0,ij}^A \sum_{l,m} \left(\frac{q_1^2}{M_{1,i}^2}\right)^l \left(\frac{q_2^2}{M_{2,j}^2}\right)^m , \tag{77}$$

Saturating the p_1^2 and p_2^2 channels by ground states of mesons under consideration we have

$$F_{0,1S\to 1S}^A = \frac{\Pi_0^{A,NR}[n,m](m_1+m_3)}{f_{1S}^{Q_1\bar{Q}_3} M_{1,1S}^2 f_{1S}^{Q_2\bar{Q}_3} M_{2,1S}^2} M_{1,1S}^n M_{2,1S}^m . \tag{78}$$

The value of $F_{0,1S\to 1S}^A$ at fixed $n=4$ is presented in Fig. 9 at

$$\begin{aligned} k_{th} &= 1.3 \text{ GeV}, \\ m_b &= 4.6 \text{ GeV}, \\ m_c &= 1.4 \text{ GeV}. \end{aligned}$$

Further, in (77) we can take into account the dominant subleading term, which is the contribution by the transition of $2S \to 2S$. In this case one could expect that the form factor is not suppressed in comparison with the contribution by the $1S \to 2S$ transition, since in the potential picture the latter decay has to be neglected because the overlapping between the wave functions at zero recoil is close to zero for the states with the different quantum numbers [6]. So, we can easily modify the relation (78) due to the second transition and justify the value of $F_{0,2S\to 2S}^A$ to reach the stability of $F_{0,1S\to 1S}^A$ at low values of moment numbers. We find $F_{0,2S\to 2S}^A / F_{0,1S\to 1S}^A \approx 3.7$ and present the behaviour of the form factor F_0^A at zero recoil in Fig. 10.

3.2 Contribution of the gluon condensate

Following a general formalism for the calculation of gluon condensate contribution in the Fock-Schwinger gauge [22, 20] we have considered diagrams, depicted in Fig. 3. In our calculations, we have used the NRQCD approximation and analysed the limit, where the invariant mass of the lepton pair takes its maximum value. Here we would like to note that the spin structure, in the leading order of relative velocity of heavy quarks, does not change, in comparison with the bare loop result for the nonrelativistic quarks. Thus, we can conclude, that in this approximation, the symmetries of NRQCD lagrangian lead to a universal Wilson coefficient for the gluon condensate operator. As a consequence, relations (69) remain valid.

Below we perform calculations for the form factor $F_0^A(p_1^2, p_2^2, Q^2)$ in the limit $q^2 \to q^2_{max}$. The contribution of gluon condensate to the corresponding correlator is given by the following expression:

$$\Delta F_0^{G^2} = \langle \frac{\alpha_s}{\pi} G_{\mu\nu}^2 \rangle \cdot \frac{\pi}{48} [3R_0 - R_2], \tag{79}$$

[6] The corresponding estimates were performed in [26], where the $1S \to 2S$ transition is suppressed with respect to $1S \to 1S$ as $1/25$.

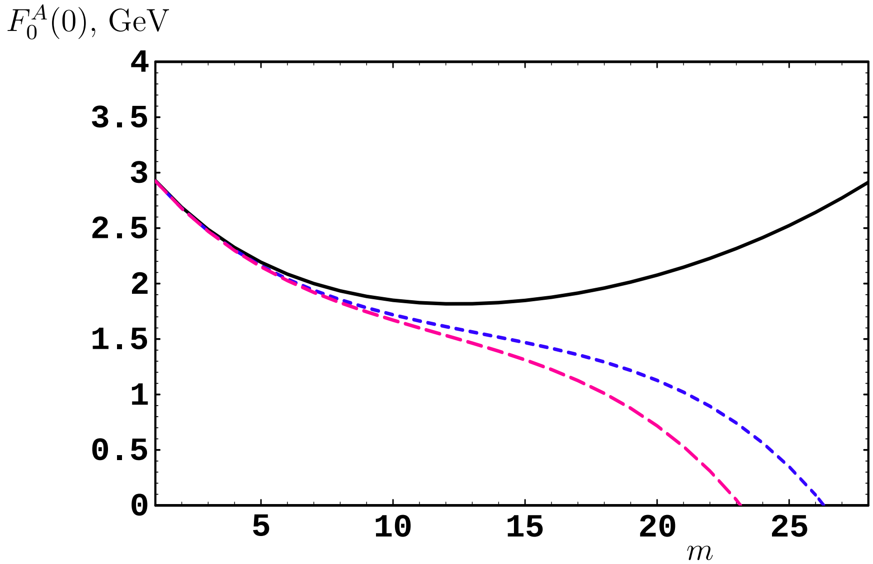

Figure 9: The NRQCD sum rule results for the $F_0^A(0)$ form factor. The solid line represents the bare quark loop contribution, the short dashed line is the result obtained by taking into account the gluon condensate term R_0 only, and the long dashed line is the form factor including the full expression for the gluon condensate.

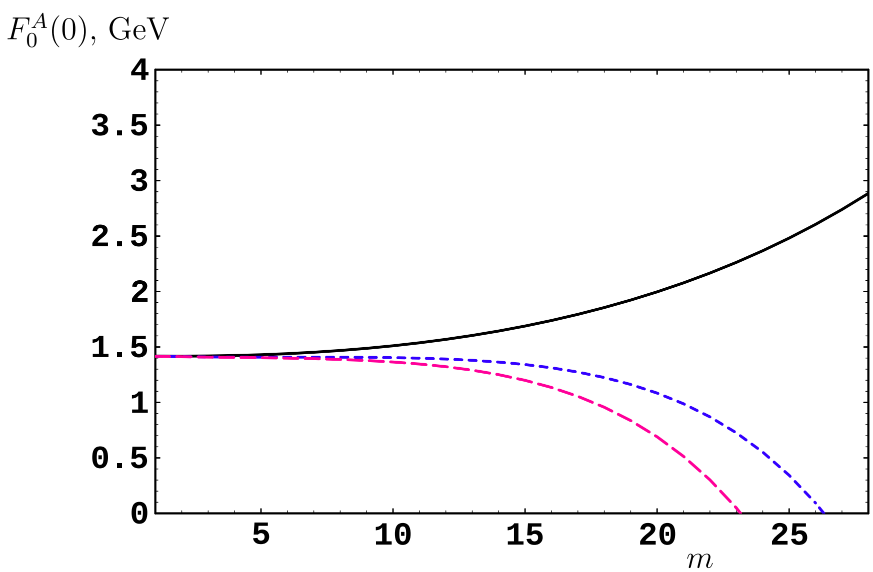

Figure 10: The NRQCD sum rule results for the $F_0^A(0)$ form factor. The contribution of $2S \to 2S$ transition has been taken into account. The notations are the same as in Fig. 9.

where

$$
\begin{aligned}
R_0 &= -\frac{1}{\pi^2 i}\int \frac{k^2 dk dk_0}{P_3(k,k_0)P_1(k,k_0,\omega_1)P_2(k,k_0,\omega_2)}R_g, \\
R_2 &= -\frac{1}{\pi^2 i}\int \frac{k^4 dk dk_0}{P_3(k,k_0)P_1(k,k_0,\omega_1)P_2(k,k_0,\omega_2)}R_k,
\end{aligned} \tag{80}
$$

In these expressions the inverse propagators are

$$
\begin{aligned}
P_1(k,k_0,\omega_1) &= \omega_1 + k_0 - \frac{k^2}{2m_1}, \\
P_2(k,k_0,\omega_2) &= \omega_2 + k_0 - \frac{k^2}{2m_2}, \\
P_3(k,k_0) &= -k_0 - \frac{k^2}{2m_3},
\end{aligned} \tag{81}
$$

The functions R_g and R_k are symmetric under the permutation of indices 1 and 2 and they have the following forms:

$$
\begin{aligned}
R_g &= -\frac{1}{m_1}\left[\frac{1}{P_3}+\frac{1}{P_1}\right]\frac{1}{P_1^2} - \frac{1}{m_2}\left[\frac{1}{P_3}+\frac{1}{P_1}\right]\frac{1}{P_2^2}, \\
R_k &= \frac{3}{m_1^2}\frac{1}{P_1^4} + \frac{3}{m_2^2}\frac{1}{P_2^4} + \frac{3}{m_1^2}\frac{1}{P_1^3 P_3} + \frac{3}{m_2^2}\frac{1}{P_2^3 P_3} + \\
&\quad \frac{1}{m_1 m_3}\frac{1}{P_1^2 P_3^2} + \frac{1}{m_2 m_3}\frac{1}{P_2^2 P_3^2} - \frac{1}{m_3^2}\frac{1}{P_3^4} - \\
&\quad \frac{1}{m_1 m_3}\frac{1}{P_1 P_3^3} - \frac{1}{m_2 m_3}\frac{1}{P_2 P_3^3} - \frac{1}{m_1 m_2}\frac{1}{P_1 P_2 P_3^2} - \frac{1}{m_1 m_2}\frac{1}{P_1^2 P_2^2}.
\end{aligned} \tag{82}
$$

Let us note, that in the calculations of diagrams in Fig. 3 we have used the following vertex for the interaction of heavy quark with the gluon

$$
L_{int}^v = -g_s \bar{h}_v v_\mu A^\mu h_v, \tag{83}
$$

where $A_\mu = A_\mu^a \cdot \frac{\lambda^a}{2}$ and its Fourier-transform in Fock-Schwinger gauge has the form

$$
A_\mu^a(k_g) = -\frac{i}{2}G_{\mu\nu}^a(0)\frac{\partial}{\partial k_g}(2\pi)^4\delta(k_g). \tag{84}
$$

After two differentiations of the nonrelativistic propagators

$$
\frac{1}{P(p,m_Q)} = \frac{1}{pv + \frac{p^2}{2m_Q}}, \tag{85}
$$

two types of contributions to the gluon condensate correction appear. The first is equal to

$$
v_\mu v_\nu \alpha_s G_{\mu\alpha}^a G_{\nu\beta}^a \cdot g^{\alpha\beta} \to \langle \alpha_s G_{\mu\nu}^2 \rangle \cdot \frac{3}{12}, \tag{86}
$$

and it leads to the term with R_0. The second expression

$$v_\mu v_\nu \alpha_s G^a_{\mu\alpha} G^a_{\nu\beta} k^\alpha k^\beta \to \langle \alpha_s G^2_{\mu\nu} \rangle \frac{k^2 - (v \cdot k)^2}{12} = -\langle \alpha_s G^2_{\mu\nu} \rangle \frac{|\vec{k}|^2}{12} \tag{87}$$

leads to the term with R_2, which is of the same order in the relative velocity of heavy quarks as R_0, but it is suppressed numerically in the region of moderate numbers for the momenta of spectral densities, where the non-perturbative contributions (condensates) of higher dimension operators are not essential.

It is easy to show, that the contributions of R_0 and R_2 can be obtained by the differentiation of two basic integrals:

$$\begin{aligned} E_0 &= -\frac{1}{\pi^2 i} \int_0^\infty \frac{k^2 dk dk_0}{P_1 P_2 P_3} = \frac{2m_{13} 2m_{23}}{k_{13} + k_{23}}, \\ E_2 &= -\frac{1}{\pi^2 i} \int_0^\infty \frac{k^4 dk dk_0}{P_1 P_2 P_3} = -\frac{2m_{13} 2m_{23}}{k_{13} + k_{23}} (k_{13}^2 + k_{13} k_{23} + k_{23}^2), \end{aligned} \tag{88}$$

where $k_{13} = \sqrt{-2m_{13}\omega_1}$, $k_{23} = \sqrt{-2m_{23}\omega_2}$. Then for R_0 and R_2 we have:

$$\begin{aligned} R_0 &= \hat{R}_g \cdot E_0, \\ R_2 &= \hat{R}_k \cdot E_2, \end{aligned}$$

where operators $\hat{R}_g$ and $\hat{R}_k$ can be obtained from R_g and R_k in (82) after the substitutions:

$$\begin{aligned} \frac{1}{P_1^n} &\to \frac{(-1)^n}{n!} \frac{\partial^n}{\partial \omega_1^n}, \\ \frac{1}{P_2^m} &\to \frac{(-1)^m}{m!} \frac{\partial^m}{\partial \omega_2^m}, \\ \frac{1}{P_3^l} &\to \frac{(-1)^l}{l!} \left[\frac{\partial}{\partial \omega_1} + \frac{\partial}{\partial \omega_2} \right]^l, \end{aligned}$$

As a result we have:

$$\begin{aligned} R_0 = & \{ \frac{1}{6m_1} \frac{\partial^3}{\partial \omega_1^3} + \frac{1}{6m_2} \frac{\partial^3}{\partial \omega_2^3} + \frac{1}{2m_1} \left(\frac{\partial^3}{\partial \omega_1^3} + \frac{\partial^3}{\partial \omega_1^2 \partial \omega_2} \right) + \\ & \frac{1}{2m_2} \left(\frac{\partial^3}{\partial \omega_2^3} + \frac{\partial^3}{\partial \omega_2^2 \partial \omega_1} \right) \} \cdot \frac{2m_{13} 2m_{23}}{k_{13} + k_{23}}, \end{aligned} \tag{89}$$

$$\begin{aligned} R_2 = & \{ \frac{3}{4! m_1^2} \frac{\partial^4}{\partial \omega_1^4} + \frac{3}{4! m_2^2} \frac{\partial^4}{\partial \omega_2^4} + \frac{3}{3! m_1^2} \left(\frac{\partial^4}{\partial \omega_1^4} + \frac{\partial^4}{\partial \omega_1^3 \partial \omega_2} \right) + \\ & \frac{1}{3! m_2^2} \left(\frac{\partial^4}{\partial \omega_2^4} + \frac{\partial^4}{\partial \omega_2^3 \partial \omega_1} \right) + \frac{1}{4 m_1 m_3} \left(\frac{\partial^4}{\partial \omega_1^4} + 2 \frac{\partial^4}{\partial \omega_1^3 \partial \omega_2} + \frac{\partial^4}{\partial \omega_1^2 \partial \omega_2^2} \right) + \\ & \frac{1}{4 m_2 m_3} \left(\frac{\partial^4}{\partial \omega_2^4} + 2 \frac{\partial^4}{\partial \omega_2^3 \partial \omega_1} + \frac{\partial^4}{\partial \omega_1^2 \partial \omega_2^2} \right) - \\ & \frac{1}{4! m_3^2} \left(\frac{\partial^4}{\partial \omega_1^4} + 4 \frac{\partial^4}{\partial \omega_1^3 \partial \omega_2} + 6 \frac{\partial^4}{\partial \omega_1^2 \partial \omega_2^2} + 4 \frac{\partial^4}{\partial \omega_1 \partial \omega_2^3} + \frac{\partial^4}{\partial \omega_2^4} \right) - \end{aligned} \tag{90}$$

$$\frac{1}{3!m_1m_3}\left(\frac{\partial^4}{\partial\omega_1^4}+3\frac{\partial^4}{\partial\omega_1^3\partial\omega_2}+3\frac{\partial^4}{\partial\omega_1^2\partial\omega_2^2}+\frac{\partial^4}{\partial\omega_1\partial\omega_2^3}\right)-$$
$$\frac{1}{3!m_2m_3}\left(\frac{\partial^4}{\partial\omega_2^4}+3\frac{\partial^4}{\partial\omega_2^3\partial\omega_1}+3\frac{\partial^4}{\partial\omega_1^2\partial\omega_2^2}+\frac{\partial^4}{\partial\omega_2\partial\omega_1^3}\right)-$$
$$\frac{1}{2m_1m_2}\left(\frac{\partial^4}{\partial\omega_1^3\partial\omega_2}+2\frac{\partial^4}{\partial\omega_1^2\partial\omega_2^2}+\frac{\partial^4}{\partial\omega_1\partial\omega_2^3}\right)-\frac{1}{4m_1m_2}\left(\frac{\partial^4}{\partial\omega_1^2\partial\omega_2^2}\right)\}\cdot$$
$$\left(-\frac{2m_{13}2m_{23}}{m_{13}+m_{23}}\right)(k_{13}^2+k_{13}k_{23}+k_{23}^2),$$

Equations (79), (89) and (90) represent the most compact analytical expression for the contribution of gluon condensate to the form factor F_0^A, whereas performing the differentiations leads to very cumbersome expressions.

In the moment scheme of sum rules we suppose

$$\begin{aligned}\omega_1 &= -(m_1+m_3)+q_1,\\ \omega_2 &= -(m_2+m_3)+q_2,\end{aligned}$$

and expand $\Delta F_0^{G^2}$ in a series over $\{q_1,q_2\}$ at the point $\{0,0\}$, which allows us to determine $\Delta F_0^{G^2}[n,m]=\frac{1}{n!}\frac{1}{m!}\frac{d^{n+m}}{dq_1^n dq_2^m}\Delta F_0^{G^2}$. Further analysis has been performed numerically with the help of MATHEMATICA. The evaluation of the $\Delta F_0^{G^2}[n,m]$ dependence in a broad range of $[n,m]$ takes too much calculation time, so we restrict ourselves by showing the results in Figs. 9 and 10 for the fixed $n=4$ with the following set of parameters

$$\begin{aligned}m_b &= 4.6\ \text{GeV},\\ m_c &= 1.4\ \text{GeV},\\ \langle\frac{\alpha_s}{\pi}G_{\mu\nu}^2\rangle &= 1.7\cdot 10^{-2}\ \text{GeV}^4.\end{aligned}$$

by taking into account (Fig. 9) and without accounting (Fig. 10) the contribution of transition $2S\to 2S$, where $F_0(2S\to 2S)/F_0(1S\to 1S)\approx 3.7$. As can be seen in these figures, the gluon contribution, while varying the moment number in the region of bound $\bar{c}c$-states, plays an important role, because it allows us to extend the stability region for the form factor F_0^A up to three times (from $m<5$ till $m<15$), and, thus, the reliability of sum rule predictions. Let us also note, that in the scheme of saturation for the hadronic part of sum rules by the ground states in both variables s_1 and s_2, the account for the gluon condensate leads to the 20% reduction for the value of form factor F_0^A.

The analysis of the dependence on the moment number m in the region of bound states $\bar{b}c$ at fixed m shows that the contribution of gluon condensate in this case does not affect the character of this dependence. This may be explained by the fact that the Coulomb corrections to the Wilson coefficient of gluon operator $G_{\mu\nu}^2$ play an essential role[7]. The summation of α_s/v-terms for $\bar{b}c$ may give a sizeable effect, contrary to the situation with the $\bar{c}c$ system, where the relative velocity of heavy quarks is not too small.

[7] At present, there is an analytical calculation of initial six moments for the Wilson coefficient of gluon condensate in the second order of α_s [27]. In this region, the influence of gluon condensate to the sum rule results is negligibly small for the heavy quarkonia, containing the b-quark, which does not allow one to draw definite conclusions on the role of such α_s-corrections.

To conclude, we have calculated the contribution of gluon condensate to the three-point sum rules for heavy quarkonia in the leading order of relative velocity of heavy quarks and in the first order of α_s. Due to the symmetry of NRQCD in the limit, when the invariant mass of the lepton pair takes its maximum value, i.e. at the recoil momentum close to zero, the Wilson coefficient for the form factor F_0^A is universal in the sense that it determines the contributions of gluon condensate to other form factors, in accordance with relations (69).

4 Conclusions

We have calculated the semileptonic decays of B_c meson in the framework of sum rules in QCD and NRQCD. We have extended the previous evaluations in QCD to the case of massive leptons: the complete set of double spectral densities in the bare quark-loop approximation have been presented. The analysis in the sum rule schemes of density moments and Borel transform has been performed and consistent results have been obtained.

We have taken into account the gluon condensate contribution for the form factors of semipeltonic transitions between the heavy quarkonia in the Borel transform sum rules of QCD, wherein the analitycal expressions for the case of three nonzero masses of quarks have been presented.

We have considered the soft limit on the form factors in NRQCD at the recoil momentum close to zero, which has allowed one to derive the generalized relations due to the spin symmetry of effective lagrangian. The relations have shown a good agreement with the numerical estimates in full QCD, which means the corrections in both the relative velocities of heavy quarks inside the quarkonia and the inverse heavy quark masses to be small within the accuracy of the method. Next, we have presented the analytical results on the gluon condensate term in the NRQCD sum rules within the moments scheme.

In both the QCD and NRQCD sum rules, the account for the gluon condensate has allowed one to enforce the reliability of predictions, since the region of physical stability for the form factors evaluated has significantly expanded in comparison with the leading order calculations of bare quark-loop contribution.

Next, we have investigated the role played by the Coulomb α_s/v corrections for the semileptonic transitions between the heavy quarkonia. We have shown that as in the case of two-point sum rules, the three-point spectral densities are enhanced due to the Coulomb renormalization of quark-meson vertices.

The complete analysis shows that the numerical estimates of various branching fractions:

$$\begin{aligned} \mathrm{BR}[B_c^+ \to J/\psi l^+ \nu] &= 2.5 \pm 0.5\%, \\ \mathrm{BR}[B_c^+ \to \bar{c}c + X] &= 23 \pm 5\%, \end{aligned}$$

agree with the results obtained in the framework of potential models and Operator Product Expansion in NRQCD. More detailed results are presented in tables.

Thus, we draw the conclusion that at present the theoretical predictions on the semileptonic decays of B_c meson give consistent and reliable results.

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5 Appendix A

In this Appendix we present the derivation of exclusive semileptonic widths for the B_c-meson decays into the J/ψ, η_c mesons with account of lepton masses.

The exclusive semileptonic width Γ_{SL} for the decay $B_c \to J/\psi(\eta_c) l\bar{\nu}_l$, where $l = e, \mu$ or τ, can be written down in the form [28]

$$\Gamma_{SL} = \frac{1}{(2\pi)^3}\frac{G_F^2|V_{cb}|^2}{M_{B_c}}\int d^4q \int d\tau_l L^{\alpha\beta}W_{\alpha\beta}, \tag{91}$$

where $d^4q = 2\pi|\vec{q}|dq^2dq_0$, $d\tau_l = |\vec{p}_l|d\Omega_l/(16\pi^2\sqrt{q^2})$ is the leptonic pair phase space, $d\Omega_l$ is the solid angle of charged lepton l, $|\vec{p}_l| = \sqrt{q^2}\Phi_l/2$ is its momentum in the dilepton center of mass system and $\Phi_l \equiv \sqrt{1-2\lambda_+ + \lambda_-^2}$, with $\lambda_\pm \equiv (m_l^2 \pm m_{\nu_l}^2)/q^2$. The tensors $L^{\alpha\beta}$ and $W_{\alpha\beta}$ in Eq. (91) are given by

$$L^{\alpha\beta} = \frac{1}{4}\sum_{spins}(\bar{l}O^\alpha\nu_l)(\bar{\nu}_lO^\beta l) = 2[p_l^\alpha p_{\nu_l}^\beta + p_l^\beta p_{\nu_l}^\alpha - g^{\alpha\beta}(p_l\cdot p_{\nu_l}) + i\epsilon^{\alpha\beta\gamma\delta}p_{l\gamma}p_{\nu_l\delta}], \tag{92}$$

and

$$\begin{aligned} W_{\alpha\beta} &= \int \frac{d^3\vec{p}_2}{2E_2}\delta^4(p_1 - p_2 - q)\tilde{W}_{\alpha\beta} \\ &= \theta(E_2)\delta(M_1^2 - 2M_1\cdot q_0 + q^2 - M_2^2)\tilde{W}_{\alpha\beta}|_{p_2=p_1-q}, \end{aligned} \tag{93}$$

where

$$\tilde{W}_{\alpha\beta} = (f_+(t)(p_1+p_2)_\alpha + f_-(t)q_\alpha)(f_+(t)(p_1+p_2)_\beta + f_-(t)q_\beta) \tag{94}$$

for the pseudoscalar particle in the final state, and

$$\begin{aligned} \tilde{W}_{\alpha\beta} = & -(iF_0^A g_{\alpha e} + iF_+^A p_{1e}(p_1+p_2)_\alpha + iF_-^A p_{1e}q_\alpha - F_V\epsilon_{\alpha eij}(p_1+p_2)^iq^j)\cdot \\ & (iF_0^A + iF_+^A p_{1k}(p_1+p_2)_\beta + iF_-^A p_{1k}q_\beta + F_V\epsilon_{\beta kmn}(p_1+p_2)^mq^n)\cdot \\ & (\sum_{polarizations}\epsilon^e\epsilon^{*k}) \end{aligned} \tag{95}$$

for the vector meson in the final state with the polarization ϵ_μ, where

$$\sum_{polarizations}\epsilon^e\epsilon^{*k} = \frac{p_2^ep_2^k}{M_2^2} - g^{ek} \tag{96}$$

M_2 is the mass of final state meson (J/ψ or η_c).

The integral over the leptonic phase space in Eq.(91) is given by

$$\int d\tau_l L^{\alpha\beta} = \frac{1}{4\pi}\frac{|\vec{p}_l|}{\sqrt{q^2}}\langle L^{\alpha\beta}\rangle, \tag{97}$$

with

$$\langle L^{\alpha\beta}\rangle = \frac{1}{4\pi}\int d\Omega_l L^{\alpha\beta} = \frac{2}{3}\{(1+\lambda_1)(q^\alpha q^\beta - g^{\alpha\beta}q^2) + \frac{3}{2}\lambda_2 g^{\alpha\beta}q^2\}, \tag{98}$$

where $\lambda_1 \equiv \lambda_+ - 2\lambda_-^2$ and $\lambda_2 \equiv \lambda_+ - \lambda_-^2$. Introducing the dimensionless kinematical variable $t \equiv q^2/m_b^2$ and integrating over q_0 the semileptonic width takes the following form:

$$\Gamma_{SL} = \frac{1}{64\pi^3}\frac{G_F^2|V_{bc}|^2 m_b^2}{M_1^2}\int_{t_{min}}^{t_{max}} dt\Phi_l(t)|\vec{q}|\langle L^{\alpha\beta}\rangle\tilde{W}_{\alpha\beta}, \tag{99}$$

where

$$|\vec{q}| = \frac{1}{2M_1}\sqrt{(M_1^2 + m_b^2 t - M_2^2)^2 - 4m_b^2 M_1^2 t}.$$

In Eq. (99) the limits of integration are given by $t_{min} = \frac{m_l^2}{m_b^2}$ and $t_{max} = \frac{1}{m_b^2}(M_1 - M_2)^2$.

Calculation of $\langle L^{\alpha\beta}\rangle\tilde{W}_{\alpha\beta}$ yields the following expressions

$$\begin{aligned}\langle L^{\alpha\beta}\rangle\tilde{W}_{\alpha\beta} \;=\; & \frac{1}{3}(3t^2 f_-^2(t)\lambda_2 m_b^4 + 6tf_-(t)f_+(t)(M_1^2 - M_2^2)\lambda_2 m_b^2 + \\ & f_+^2(t)(t^2(2\lambda_1 - 3\lambda_2 + 2)m_b^4 - 2t(M_1^2 + M_2^2)(2\lambda_1 - 3\lambda_2 + 2)m_b^2 + \\ & 2(M_1^2 - M_2^2)^2(\lambda_1 + 1)))\end{aligned}$$

for the pseudoscalar meson in the final state, and

$$\begin{aligned}\langle L^{\alpha\beta}\rangle\tilde{W}_{\alpha\beta} \;=\; & \frac{1}{12M_2}(2(t^2(\lambda_1+1)m_b^4 - 2t((1+\lambda_1)M_1^2 - 5(1+\lambda_1)M_2^2 + 9M_2^2\lambda_2)m_b^2 + \\ & (M_1^2 - M_2^2)^2(1+\lambda_1))(F_0^A)^2 - 2(t^2 m_b^4 - 2t(M_1^2 + M_2^2)m_b^2 + \\ & (M_1^2 - M_2^2)^2)(F_+^A(2(1+\lambda_1)tm_b^2 - 3t\lambda_2 m_b^2 - 2(1+\lambda_1)M_1^2 + \\ & 2(1+\lambda_1)M_2^2) - 3tF_-^A m_b^2\lambda_2)F_0^A + (t^2 m_b^4 - 2t(M_1^2 + M_2^2)m_b^2 + \\ & (M_1^2 - M_2^2)^2)((2t^2 m_b^4 + 2t^2\lambda_1 m_b^4 - 3t^2\lambda_2 m_b^4 - 4(1+\lambda_1)tM_1^2 m_b^2 - \\ & 4(1+\lambda_1)tM_2^2 + 6tM_1^2\lambda_2 m_b^2 + 6tM_2^2\lambda_2 m_b^2 + 2(1+\lambda_1)M_1^4 + \\ & 2(1+\lambda_1)M_2^4 - 4(1+\lambda_1)M_1^2 M_2^2)(F_+^A)^2 + \\ & 6tF_-^A F_+^A\lambda_2 m_b^2(M_1^2 - M_2^2) + tm_b^2(3t(F_-^A)^2\lambda_2 m_b^2 + \\ & 16(1+\lambda_1)(F_V)^2 M_2^2 - 24\lambda_2 M_2^2(F_V)^2)))\end{aligned}$$

for the vector meson in the final state.

6 Appendix B

In this Appendix we illustrate the kind of expressions, which arise for the gluon condensate contribution to the form factors $F^i(t)$ in the framework of Borel transformed three point sum rules, in the case of $f_1(t) = f_+(t) + f_-(t)$ form factor.

Following the algorithm, described in section **2.3**, for $\Pi_1^{\langle G^2\rangle}$ we have the following expression

$$\Pi_1^{\langle G^2\rangle} = C_1^{(-1,-2)}U_0(-1,-2)+C_1^{(-1,-1)}U_0(-1,-1)+\sum_{i=-4}^{0}C_1^{(0,i)}U_0(0,i)+ \sum_{i=-5}^{-1}C_1^{(1,i)}U_0(1,i)+\sum_{i=-6}^{0}C_1^{(2,i)}U_0(2,i)+\sum_{i=-6}^{0}C_1^{(3,i)}U_0(3,i), \tag{100}$$

where

$$C_1^{(-1,-2)} = -\frac{M_{bc}^2+M_{cc}^2}{12M_{bc}^2}, \tag{101}$$

$$C_1^{(-1,-1)} = -\frac{1}{12M_{bc}^2}, \tag{102}$$

$$C_1^{(0,-4)} = \frac{1}{48M_{bc}^6M_{cc}^4}(M_{bc}^{10}(6m_2^2+m_3^2)+8m_2^2M_{bc}^8M_{cc}^2+(-5m_1^2+5m_2^2+ m_3^2)M_{bc}^6M_{cc}^4+(-7m_1^2+3m_2^2+2m_1(m_2-m_3))M_{bc}^4M_{cc}^6-4m_1^2M_{bc}^2M_{cc}^8 -2m_1^2M_{cc}^{10}), \tag{103}$$

$$C_1^{(0,-3)} = \frac{1}{48M_{bc}^6M_{cc}^4}((m_1^2-2m_1m_3-2m_3^2)M_{cc}^8+(6m_2^2+4M_{cc}^2)M_{bc}^8- (2m_1^2+2m_2^2+6m_2m_3+2m_3^2+2m_1(-5m_2+m_3)-11M_{cc}^2 -2Q^2)M_{bc}^6M_{cc}^2+(m_1^2-m_2^2+2m_1(m_2-5m_3)-2m_2m_3+2m_3^2 +5M_{cc}^2+Q^2)M_{bc}^2M_{cc}^6+(-7m_1^2+4m_1m_2-2m_2^2-10m_1m_3- 8m_2m_3+2m_3^2+12M_{cc}^2+2Q^2)M_{bc}^4M_{cc}^4), \tag{104}$$

$$C_1^{(0,-2)} = \frac{1}{48M_{bc}^6M_{cc}^4}((3m_1^2-2m_1m_3-2m_3^2)M_{cc}^6+(m_2^2-5m_3^2+9M_{cc}^2)M_{bc}^6+ (-4m_1^2-7m_2^2+2m_1(6m_2-5m_3)-8m_2m_3-13m_3^2+16M_{cc}^2+ 4Q^2)M_{bc}^4M_{cc}^2-(5m_1^2-4m_1(m_2-4m_3)+2(m_2^2+2m_2m_3+m_3^2- M_{cc}^2-Q^2)M_{bc}^2M_{cc}^4), \tag{105}$$

$$C_1^{(0,-1)} = \frac{1}{48M_{bc}^6M_{cc}^4}(-2m_3^2M_{cc}^4+(-6m_3^2+5M_{cc}^2)M_{bc}^4+ (-m_1^2+2m_1m_2-m_2^2+4m_2m_3-16m_3^2+M_{cc}^2+Q^2)M_{bc}^2M_{cc}^2, \tag{106}$$

$$C_1^{(0,0)} = \frac{m_3^2(M_{bc}^2+M_{cc}^2)}{48M_{bc}^6M_{cc}^4}, \tag{107}$$

$$C_1^{(1,-5)} = -\frac{1}{48M_{bc}^8M_{cc}^6}(m_2^2M_{bc}^4-m_1^2M_{cc}^4)((5m_2^2+m_3^2)M_{bc}^8-m_3^2M_{bc}^6M_{cc}^2- (4m_1^2+3m_2^2)M_{bc}^4M_{cc}^4+2m_1(m_2-m_3)M_{bc}^2M_{cc}^6+2m_1^2M_{cc}^4), \tag{108}$$

$$C_1^{(1,-4)} = \frac{1}{48M_{bc}^8M_{cc}^6}(m_1^2(3m_1^2+2m_1m_3+6m_3^2)M_{cc}^{10}+ (-2m_2^4-7m_2^2M_{cc}^2+m_3^2M_{cc}^2)M_{bc}^{10}+(8m_2^4+4m_2^3m_3+3m_2^2m_3^2+ 2m_2m_3^3+m_1^2(4m_2^2+m_3^2)-2m_1(4m_2^3-3m_2^2m_3+m_2m_3^2-m_3^3- 11m_2^2M_{cc}^2-3m_3^2M_{cc}^2-4m_2^2Q^2-m_3^2Q^2)M_{bc}^8M_{cc}^2-(3m_1^4+6m_2^2m_3^2+ \tag{109}$$

$$m_1^3(-6m_2+4m_3)+2m_1(m_2^2m_3+2m_2M_{cc}^2-2m_3M_{cc}^2)+ m_1^2(10m_2^2+2m_2m_3+2m_3^2-8M_{cc}^2-3Q^2))M_{bc}^4M_{cc}^6+(4m_1m_2^2m_3+ m_1^2(9m_2^2+16M_{cc}^2)+m_2^2(2m_2^2+4m_2M-3-4m_3^2-M_{cc}^2- 2Q^2))M_{bc}^6M_{cc}^4-m_1(6m_1^3+4m_1^2m_3+2m_3^2(-m_2+m_3)+m_1(m_2^2+ 2m_2m_3-4m_3^2+5M_{cc}^2-Q^2))M_{bc}^2M_{cc}^8),$$

$$C_1^{(1,-3)} = \frac{1}{48M_{bc}^8M_{cc}^6}(2m_3(m_1^3+5m_1^2m_3+2m_3^3)M_{cc}^8+(m_3^4+m_2^2(9m_3^2-4M_{cc}^2+ 8m_2m_3M_{cc}^2-3m_3^2M_{cc}^2-11M_{cc}^4)M_{bc}^8+(2m_2^4+10m_2^2m_3^2+m_3^4+ 7m_2^2M_{cc}^2+18m_2m_3M_{cc}^2+2m_3^2M_{cc}^2-20M_{cc}^2-4m_1(m_2-m_3)(m_2^2 +2M_{cc}^2)+m_1^2(2m_2^2+5M_{cc}^2)-2m_2^2Q^2-5M_{cc}^2Q^2)M_{bc}^6M_{cc}^2+(-7m_2^2m_3^2 -4m_3^4+4m_2^2M_{cc}^2+10m_2m_3M_{cc}^2-7M_{cc}^4+2m_1^2(2m_2m_3-m_3^2+5M_{cc}^2)+ m_1(6m_2-6m_3-2m_2^2m_3+4m_2m_3^2+4m_2M_{cc}^2+10m_3M_{cc}^2)+2m_3^2Q^2 -4M_{cc}^2Q^2)M_{bc}^4M_{cc}^6+(m_1^4-2m_1^3(m_2-4m_3)+10m_3^2M_{cc}^2+2m_1m_3(m_2^2+ m_2m_3+m_3^2-M_{cc}^2-Q^2)+m_1^2(m_2^2+2m_2m_3+7M_{cc}^2-Q^2)), \tag{110}$$

$$C_1^{(1,-2)} = \frac{1}{48M_{bc}^8M_{cc}^6}(m_3^2(3m_1^2-2m_1m_3+6m_3^2)M_{cc}^6+(14m_2m_3M_{cc}^2+(16m_3^2- 13M_{cc}^2)M_{cc}^2+m_2^2(4m_3^2+M_{cc}^2))M_{bc}^6+(-5m_2^2m_3^2-2m_2m_3^3-2m_3^4+ 4m_2^2M_{cc}^2+8m_2m_3M_{cc}^2+32m_3^2M_{cc}^2-8M_{cc}^4+m_1^2(-3m_3^2+4M_{cc}^2)+ 2m_1(3m_2m_3^2-2m_3^3-2m_2M_{cc}^2+6m_3M_{cc}^2)+3m_3^2Q^2-4M_{cc}^2Q^2)M_{bc}^4M_{cc}^2 -(2m_1^3m_3+m_1^2(-4m_2m_3+4m_3^2+M_{cc}^2)+m_3^2(m_2^2+2m_2m_3+4m_3^2 -7M_{cc}^2-Q^2)+2m_1m_3(m_2^2-m_2m_3+3m_3^2-2M_{cc}^2-Q^2))M_{bc}^2M_{cc}^6), \tag{111}$$

$$C_1^{(1,-1)} = -\frac{1}{48M_{bc}^8M_{cc}^6}(2m_1m_3^3M_{cc}^4+(4m_3^4-17m_3^2M_{cc}^2+M_{cc}^4)M_{bc}^4+ 2m_3^2(m_1^2-2m_1m_2+m_2^2+2m_1m_3+4m_3^2-6M_{cc}^2-Q^2)), \tag{112}$$

$$C_1^{(2,-6)} = \frac{(M_{bc}^2-M_{cc}^2)(m_2^2M_{bc}^4-m_1^2M_{cc}^4)^3}{96M_{bc}^{10}M_{cc}^8}, \tag{113}$$

$$C_1^{(2,-5)} = -\frac{1}{96M_{bc}^8M_{cc}^6}(m_2^2M_{bc}^4-m_1^2M_{cc}^4)((m_2^2+2m_3^2)M_{bc}^6+7m_1^2M_{bc}^2M_{cc}^4+ m_2^2(m_1^2-2m_1m_2+m_2^2+2m_1m_3+2m_2m_3+3M_{cc}^2-Q^2)M_{bc}^4- m_1^2(m_1^2-2m_1m_2+m_2^2+2m_1m_3+2m_2m_3+7M_{cc}^2-Q^2)M_{cc}^4), \tag{114}$$

$$C_1^{(2,-4)} = \frac{1}{96M_{bc}^{10}M_{cc}^8}(-3m_1^4m_3^2M_{cc}^{10}+m_1^3(6m_2M_{bc}^2-6m_3M_{bc}^2+m_1m_3^2- 2m_1M_{cc}^2-4m_3M_{cc}^2)M_{bc}^2M_{cc}^8-(12m_2^3m_3M_{cc}^2+4m_2m_3^3M_{cc}^2-12m_2^2M_{cc}^4 -4m_3^2M_{cc}^4+m_2^4(3m_3^2+4M_{cc}^2))M_{bc}^{10}+2m_1(-3m_2^3M_{bc}^2+4m_1m_2m_3M_{cc}^2 -8m_1M_{cc}^4+m_2^2(3m_3M_{bc}^2-m_1m_3^2+3m_1M_{cc}^2+2m_3M_{cc}^2))M_{bc}^6M_{cc}^4+ 2m_1^2(-4m_2(m_1-2m_3)M_{cc}^2+m_2^2(m_3^2+3M_{cc}^2)+(3m_1^2+6m_1m_3+ 2M_{cc}^2-3Q^2)M_{cc}^2)M_{bc}^4M_{cc}^6+m_2^2(m_2^2(m_3^2-4M_{cc}^2)+4m_2(m_1-4m_3)M_{cc}^2 +2(-2m_1^2-4m_1m_3+3M_{cc}^2+2Q^2)M_{cc}^2)M_{bc}^8M_{cc}^2, \tag{115}$$

$$
\begin{aligned}
C_1^{(2,-3)} &= \frac{1}{48M_{bc}^8M_{cc}^6}(m_1(9m_3M_{bc}^2 - m_1m_3^2 + m_2(-9M_{bc}^2 - 4m_1m_3 + 4m_3^2) + \\
&\quad 4m_1M_{cc}^2)M_{bc}^2M_{cc}^4 - (2m_2^3m_3 + 9m_2^2m_3^2 + m_3^4 + 10m_2m_3M_{cc}^2 - \\
&\quad 10M_{cc}^4)M_{bc}^6 + (m_2^4m_3^2 + 2m_2^3m_3^3 - 3m_2^2m_3^2M_{cc}^2 - 3m_2^2M_{cc}^4 - 14m_2m_3M_{cc}^4 \\
&\quad +9M_{cc}^6 + m_1^2(m_2^2m_3^2 + 2m_3^2M_{cc}^2 - 3M_{cc}^4) + 2m_1(-m_2^3m_3^2 + m_2^2m_3^3 + \\
&\quad m_2^2m_3M_{cc}^2 - 2m_2m_3^2M_{cc}^2 + m_2M_{cc}^4 - 3m_3M_{cc}^4) - m_2^2m_3^2Q^2 - 2m_3^2M_{cc}^2Q^2 + \\
&\quad 3M_{cc}^2Q^2)M_{bc}^4 + m_1(m_1^3m_3^2 + 3(m_2 - m_3)m_3^2M_{bc}^2 + m_1^2(3m_2M_{bc}^2 - 3m_3M_{bc}^2 \\
&\quad -2m_2m_3^2 + 2m_3^3 - 2m_3M_{cc}^2) + m_1m_3^2(m_2^2 + 2m_2m_3 - 3M_{cc}^2 - Q^2))M_{cc}^4), \qquad (116)\\
C_1^{(2,-2)} &= \frac{1}{96M_{bc}^{10}M_{cc}^8}(3m_1^2m_3^4M_{cc}^6 + m_1m_3^2M_{bc}^2M_{cc}^4(6m_2M_{bc}^2 - 6m_3M_{bc}^2 + m_1m_3^2 - \\
&\quad 2m_1M_{cc}^2 + 4m_3M_{cc}^2) + M_{bc}^6(3m_2^2m_3^4 + 8m_2m_3(m_3^2 - M_{cc}^2)M_{cc}^2 + \\
&\quad 4M_{cc}^4(-11m_3^2 + 2M_{cc}^2)) + M_{bc}^4m_3M_{cc}^2(-8m_2(m_1 - 2m_3)m_3M_{cc}^2 + \\
&\quad m_2^2(m_3^3 + 6m_3M_{cc}^2) - 2M_{cc}^2(-3m_1^2m_3 - 6m_1m_3^2 + 2m_1M_{cc}^2 + \\
&\quad 20m_3M_{cc}^2 + 3m_3Q^2))), \qquad (117)\\
C_1^{(2,-1)} &= \frac{1}{96M_{bc}^8M_{cc}^6}m_3^2(M_{bc}^2(4m_2m_3 + 15m_3^2 - 16M_{cc}^2) + m_3(-m_1^2m_3 + \qquad (118)\\
&\quad 2m_1(m_2m_3 - m_3^2 + 2M_{cc}^2) + m_3(-m_2^2 - 2m_2m_3 + 13M_{cc}^2 + Q^2))),\\
C_1^{(2,0)} &= -\frac{m_3^4(M_{bc}^2(m_3^2 - 4M_{cc}^2) + m_3^2M_{cc}^2}{96M_{bc}^{10}M_{cc}^8}, \qquad (119)\\
C_1^{(3,-6)} &= \frac{(M_{bc}^4m_2^2 - m_1^2M_{cc}^2)^3}{96M_{bc}^{10}M_{cc}^8}, \qquad (120)\\
C_1^{(3,-5)} &= \frac{1}{96M_{bc}^8M_{cc}^8}(M_{bc}^4m_2^2 - m_1^2M_{cc}^4)(M_{bc}^4m_2^2(2m_2m_3 + M_{cc}^2) - \\
&\quad m_1^2M_{cc}^4(2m_2m_3 + 7M_{cc}^2)), \qquad (121)\\
C_1^{(3,-4)} &= -\frac{1}{96M_{bc}^{10}M_{cc}^8}(-3m_1^4m_3^2M_{cc}^8 + M_{bc}^8m_2^2(m_2^2m_3^2 - 8m_2m_3M_{cc} + 2M_{cc}^4) + \\
&\quad 2M_{bc}^4m_1^2M_{cc}^2(m_2^2m_3^2 + 6m_2m_3M_{cc}^2 + 3M_{cc}^4)), \qquad (122)\\
C_1^{(3,-3)} &= \frac{1}{48M_{bc}^8M_{cc}^8}(m_1^2m_3^2M_{cc}^4(-2m_2m_3 + M_{cc}^2) + M_{bc}^4(-2m_2^3m_3^3 + 2m_2^2m_3^2M_{cc}^2 + \\
&\quad 6m_2m_3M_{cc}^4 - 3M_{cc}^6)), \qquad (123)\\
C_1^{(3,-2)} &= -\frac{m_3^2(3m_1^2m_3^2M_{cc}^4 + M_{bc}^4(m_2^2m_3^2 + 12m_2m_3M_{cc}^2 - 18M_{cc}^4))}{96M_{bc}^{10}M_{cc}^8}, \qquad (124)\\
C_1^{(3,-1)} &= \frac{m_3^4(2m_2M - 3 - 9M_{cc}^2)}{96M_{bc}^8M_{cc}^8}, \qquad (125)\\
C_1^{(3,0)} &= \frac{m_3^6}{96M_{bc}^{10}M_{cc}^8}. \qquad (126)
\end{aligned}
$$

For functions $U_0(a,b)$ we have the following expressions

$$
U_0(a,b) = \sum_{n=1+b}^{1+a+b} 2C_{n-b-1}^a \exp[-B_0](M_{bc}^2 + M_{cc}^2)^{a+b+1-n}\left(\frac{B_{-1}}{B_1}\right)^{\frac{n}{2}} K_{-n}[2\sqrt{B_{-1}B_1}],
$$

for $a \geq 0$. Here $K_n[z]$ is the modified Bessel function of the second order. In the case of $U_0(-1,-2)$ and $U_0(-1,-1)$ we have failed to obtain exact analytical expressions, so we present their analytical approximations:

$$\begin{aligned}
U_0(-1,-2) &= \frac{\exp[-B_0]}{2v^3 B_{-1} B_1}\{2\exp[-\sqrt{\frac{B_{-1}}{B_1}}]B_1 v^2 + 2\exp[-B_{-1}^{1/2}B_1^{3/2}]\sqrt{B_{-1}B_1}v^2 \\
&+ \exp[-B_{-1}^{1/2}B_1^{3/2}]B_{-1}^{3/2}B_1^{1/2}(2+vB_1)v - \\
&2v^2 B_{-1}B_1^2\left(\exp[\frac{B_{-1}}{v}]\Gamma\left(0,\frac{B_{-1}}{v}+\sqrt{\frac{B_{-1}}{B_1}}\right)+\Gamma(0,B_{-1}^{1/2}B_1^{3/2})\right) \\
&-2vB_{-1}B_1(\exp[\frac{B_{-1}}{v}]\Gamma\left(0,\frac{B_{-1}}{v}+\sqrt{\frac{B_{-1}}{B_1}}\right)+\Gamma(0,B_{-1}^{1/2}B_1^{3/2}) \\
&-\exp[vB_1]\Gamma(0,B_{-1}^{1/2}B_1^{3/2}+vB_1)) - v^2 B_{-1}\exp[B_{-1}^{1/2}B_1^{3/2}] \qquad (127) \\
&-B_{-1}^2 B_1(v^2 B_1^2\Gamma(0,B_{-1}^{1/2}B_1^{3/2}) + 2vB_1\Gamma(0,B_{-1}^{1/2}B_1^{3/2}) \\
&+2\Gamma(0,B_{-1}^{1/2}B_1^{3/2} - 2\exp[vB_1]\Gamma(0,B_{-1}^{1/2}B_1^{3/2}+vB_1))\}, \\
U_0(-1,-1) &= \frac{-\exp[B_0]}{v^2}\{(\exp\frac{B_{-1}}{v}\Gamma\left(0,\frac{B_{-1}}{v}+\sqrt{\frac{B_{-1}}{B_1}}\right)+ \\
&\Gamma(0,B_{-1}^{1/2}B_1^{3/2}) - \exp vB_1\Gamma(0,B_{-1}^{1/2}B_1^{3/2}+vB_1)+ \\
&v(\exp\frac{B_{-1}}{v}\Gamma\left(0,\frac{B_{-1}}{v}+\sqrt{\frac{B_{-1}}{B_1}}\right) - \Gamma\left(0,\sqrt{\frac{B_{-1}}{B_1}}\right))B_1) \qquad (128) \\
&-\sqrt{\frac{B_{-1}}{B_1}}v\exp[-B_{-1}^{1/2}B_1^{3/2}] + B_{-1}(vB_1\Gamma(0,B_{-1}^{1/2}B_1^{3/2})+ \\
&\Gamma(0,B_{-1}^{1/2}B_1^{3/2}) - \exp[vB_1]\Gamma(0,B_{-1}^{1/2}B_1^{3/2}+vB_1))\},
\end{aligned}$$

where $v = M_{bc}^2 + M_{cc}^2$ and $\Gamma(a,z)$ is the incomplete gamma function, which is given by the integral $\Gamma(a,z) = \int_z^\infty t^{a-1}e^{-t}dt$.

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